RIEMANNIAN SYMMETRIC SPACES OF THE NON-COMPACT TYPE: DIFFERENTIAL GEOMETRY

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1. Introduction

Many of the rigidity questions in non-positively curved geometries that will be addressed in the more advanced lectures of this summer school either directly concern symmetric spaces or originated in similar questions about such spaces.

This course is meant to provide a quick introduction to symmetric spaces of the non-compact type, from the (differential) geometer point of view. A complementary algebraic introduction is given in P.-E. Paradan's lecture [P]. We have tried to always start from (and stick to) geometric notions, even when the aim was to obtain more algebraic results. Since the general topic of the summer school is non-positively curved geometries, we have insisted on the aspects of non-positive curvature which can be generalized to much more general settings than Riemannian manifolds, such as CAT(0)-spaces.

Of course this course is very incomplete and the reader should consult the references given at the end of the paper for much more detailed expositions of the subject.

In what follows, (M, g) denotes a (smooth and connected) Riemannian manifold of dimension n.

2. Riemannian preliminaries

In this section we review very quickly and without proofs the basics of Riemannian geometry that will be needed in the rest of the paper. Proofs and details can be found in standard text books, for example [dC], [GHL] or [KN].

2.1. Levi-Civitá connection.

A *connection* on the tangent bundle TM of M is a bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

such that, for every function $f \in C^\infty(M)$ and all vector fields $X, Y \in \Gamma(TM)$,

- $\nabla_{fX} Y = f\nabla_X Y$,
- $\nabla_X fY = \mathrm{d}f(X)Y + f\nabla_X Y$ (Leibniz rule).

Note that the value of $\nabla_X Y$ at a point m of M depends only on the value of X at m.

On a Riemannian manifold (M, g), there is a unique connection on the tangent bundle, the so-called *Levi-Civitá connection* of g, which is both torsion-free and metric, namely, such that

- $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(TM)$,
- $\nabla g = 0$, i.e. $X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \Gamma(TM)$.

The following formula for the Levi-Civitá connection, which also implies its existence, is useful:

$$2g(\nabla_X Y, Z) = X.g(Y, Z) + Y.g(X, Z) - Z.g(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).$$

2.2. Curvatures.

If $X, Y, Z \in \Gamma(TM)$, we define $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. In fact, the value of this vector field at a point m depends only on the values of the vectors fields X, Y, Z at m. R is called the *Riemann curvature tensor* of g.

The metric allows us to see the Riemann curvature tensor as a (4,0)-tensor by setting $R(X,Y,Z,T) = g(R(X,Y)Z,T)$

The Riemann curvature tensor has the following symmetries [GHL, Proposition 3.5]:

- $R(X,Y,Z,T) = -R(Y,X,Z,T) = R(Z,T,X,Y)$,
- First Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

The *sectional curvature* $K(P)$ of a 2-plane P in T_mM is defined as follows : pick a g-orthonormal basis (u,v) of P and set $K(P) = R(u,v,u,v)$. The sectional curvature coincides with the usual notion of Gaussian curvature on a surface. Namely, if P is a tangent 2-plane in T_mM and S a small piece of surface in M tangent to P at m, then the sectional curvature of P is the Gaussian curvature of S at m [dC, p. 130-133].

Note that the sectional curvatures determine the curvature tensor [dC, p. 94].

2.3. Parallel transport, geodesics and the exponential map.

The Levi-Civitá connection allows to differentiate vector fields defined along curves[GHL, Theorem 2.68]. If c is a curve in M and X a vector field along c, we call $\nabla_{\dot c}X$, or X' when no confusion is possible, the *covariant derivative* of X along c: it is a new vector field along c.

A vector field X along a curve c is called *parallel* if its covariant derivative along c vanishes identically: $\nabla_{\dot c}X = 0$. It follows from the standard theory of differential equations that given a curve c and a vector v tangent to M at $c(0)$, there exists a unique parallel vector field X_v along c such that $X_v(0) = v$. The *parallel transport* along c from $c(0)$ to $c(t)$ is by definition the linear isomorphism given by $v \in T_{c(0)}M \mapsto X_v(t) \in T_{c(t)}M$. Since ∇ is metric, the parallel transport is in fact a linear isometry $T_{c(0)}M \longrightarrow T_{c(t)}M$ [GHL, Proposition 2.74].

A *geodesic* is a smooth curve $\gamma : I \longrightarrow M$ such that $\nabla_{\dot\gamma}\dot\gamma = 0$.

Note that a geodesic always has constant speed [GHL, 2.77].

One can prove (see for example [dC, p. 62-64]) that given a point m in M and a tangent vector $v \in T_mM$, there exist $\varepsilon > 0$ and a geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = m$ and $\dot\gamma(0) = v$. This geodesic is unique, depends in a C^∞ way of m and v. It will generally be denoted γ_v (or σ_v).

Proposition 2.1. [dC, p. 64] *For all $m \in M$, there exists a neighborhood $\mathcal{U}$ of m and $\delta > 0$ such that, for all $x \in \mathcal{U}$ and all $v \in T_xM$ with $\|v\| < \delta$, the geodesic γ_v is defined on $]-2,2[$.*

Let $x \in M$. The *exponential map* at x is the map $\exp_x : v \in T_xM \mapsto \gamma_v(1) \in M$, defined on a sufficiently small neighborhood of 0 in T_xM.

The differential at $0 \in T_xM$ of $\exp_x$ is the identity map and therefore:

Theorem 2.2. [dC, p. 65] *For all $x \in M$, there exists $\delta > 0$ such that the restriction of $\exp_x : T_xM \longrightarrow M$ to the ball $B(0,\delta)$ is a diffeomorphism onto its image.*

A neighborhood $\mathcal{U}$ of $m \in M$ is called a *normal neighborhood* of m if it is the diffeomorphic image under $\exp_m$ of a star-shaped neighborhood of $0 \in T_mM$.

Theorem 2.3. [dC, p. 72 & 76] *Each $m \in M$ has a normal neighborhood $\mathcal{U}_m$ which is also a normal neighborhood of each of its points. In particular, any two points of $\mathcal{U}_m$ can be joined by a unique geodesic in $\mathcal{U}_m$.*

Such a neighborhood will be called a *convex normal neighborhood* of m.

2.4. **Jacobi fields. Differential of the exponential map.**

Let γ be a geodesic in M. A vector field Y along γ is called a *Jacobi vector field* if it satisfies the differential equation along γ:

$$Y'' + R(\dot\gamma, Y)\dot\gamma = 0.$$

This equation is equivalent to a linear system of ordinary second order linear equations ([dC, p. 111]) and therefore for any $v, w \in T_{\gamma(0)}M$, there exists a unique Jacobi vector field Y such that $Y(0) = v$ and $Y'(0) = w$. The space $J(\gamma)$ of Jacobi vector fields along γ is $2n$-dimensional.

Note that $t \mapsto \dot\gamma(t)$ and $t \mapsto t\dot\gamma(t)$ are Jacobi vector fields along γ. If Y is a Jacobi field along γ such that $Y(0)$ and $Y'(0)$ are orthogonal to $\dot\gamma(0)$ then $Y(t)$ is orthogonal to $\dot\gamma(t)$ for all t (such Jacobi fields are called *normal Jacobi fields*).

Let H be a variation of geodesics. This means that H is a differentiable map from a product $I \times J$ into M such that for all s the curve $t \mapsto \gamma_s(t) := H(s,t)$ is a geodesic in M. It is then easy to see that the vector field Y along γ_0 given by $Y(t) = \frac{\partial H}{\partial s}(0,t)$ is a Jacobi vector field [GHL, 3.45].

In particular, we obtain an explicit formula for Jacobi fields along $t \mapsto \gamma(t)$ vanishing at $t = 0$ in terms of the exponential map. Indeed, for any $v, w \in T_mM$, the derivative Y of the variation of geodesic $H(s,t) = \exp_m(t(v + sw))$ is a Jacobi vector field along the geodesic $\gamma : t \mapsto H(0,t) = \exp_m(tv)$. But $Y(t) = \mathrm{d}_{tv}\exp_m(tw)$ and

$$Y'(t) = \nabla_{\dot\gamma}(t\mathrm{d}_{tv}\exp_m(w)) = \mathrm{d}_{tv}\exp_m(w) + t\nabla_{\dot\gamma}\mathrm{d}_{tv}\exp_m(w)$$

so that $Y'(0) = \mathrm{d}_0\exp_m(w) = w$. From uniqueness, we obtain:

Proposition 2.4. [dC, p. 114] *Let $t \mapsto \gamma(t) = \exp_m(tv)$ be a geodesic in M. Then any Jacobi vector field Y along γ such that $Y(0) = 0$ is given by $Y(t) = \mathrm{d}_{tv}\exp_m(tY'(0))$.*

2.5. **Riemannian manifolds as metric spaces.**

The *length* of a (piecewise) differentiable curve $c : [a,b] \longrightarrow M$ is defined to be

$$L(c) = \int_a^b \|\dot c(t)\|_g \, dt.$$

A curve c is a geodesic if and only if it locally minimizes length, meaning that for all t, there exists ε such that c is the shortest curve between $c(t-\varepsilon)$ and $c(t+\varepsilon)$ (see for example [GHL, p. 91]).

A geodesic is called *minimizing* if it minimizes length between any two of its points.

Given two points x and y of M, define $d(x,y)$ to be the infimum of the length of all piecewise differentiable curves joining x to y. Then d defines a distance on M compatible with the manifold topology of M [GHL, p. 87]. We call it the *length metric* of (M,g).

We have the following very important theorem (for a proof see [dC, p. 146] or [GHL, p. 94]):

Theorem 2.5 (Hopf-Rinow). *Let (M,g) be a Riemannian manifold. The following assertions are equivalent:*

(1) *M is geodesically complete, namely, all the geodesics are defined over $\mathbb{R}$, or equivalently, for all $m \in M$, $\exp_m$ is defined on T_mM;*

(2) *There exists $m \in M$ such that $\exp_m$ is defined on T_mM;*

(3) *(M,d) is complete as a metric space;*

(4) *the closed bounded subsets of M are compact.*

Moreover, all these assertions imply that given any two points in M, there exists a minimizing geodesic joining them.

We can also give a metric interpretation of sectional curvature showing that it gives a measurement of the rate at which geodesics infinitesimally spread apart:

Proposition 2.6. [C] *Let u and v be two orthonormal tangent vectors at $m \in M$. Let σ_u and σ_v be the corresponding unit speed geodesics. Call κ the sectional curvature of the 2-plane spanned by u and v. Then*

$$d(\sigma_u(t), \sigma_v(t))^2 = 2t^2 - \frac{\kappa}{6}t^4 + o(t^5).$$

2.6. **Isometries.**

A map $f : M \longrightarrow M$ is a *local isometry* of (M, g) if for all $x \in M$, $\mathrm{d}_x f$ is a (linear) isometry: $\forall u, v \in T_x M$, $g_{f(x)}(\mathrm{d}_x f(u), \mathrm{d}_x f(v)) = g_x(u, v)$. Note that a local isometry is necessarily a local diffeomorphism.

A local isometry is called an *isometry* if it is a global diffeomorphism.

An isometry of (M, g) maps geodesics to geodesics and is therefore an *affine transformation* of M. It is also obviously a distance preserving map of the metric space (M, d).

Conversely, one can prove :

Theorem 2.7. [H, p. 61] *Let (M, g) be a Riemannian manifold. Then:*

(1) *Any affine transformation f such that $\mathrm{d}_x f$ is isometric for some $x \in M$ is an isometry of M.*

(2) *Any distance preserving map of the metric space (M, d) onto itself is an isometry of M.*

One also has the useful

Lemma 2.8. [dC, p. 163] *Let ϕ and ψ be two isometries of M. Assume that at some point x, $\phi(x) = \psi(x)$ and $\mathrm{d}_x\phi = \mathrm{d}_x\psi$. Then $\phi = \psi$.*

and the

Proposition 2.9. [GHL, p. 96] *Let $f : M \longrightarrow N$ be a local isometry between two Riemannian manifolds. Assume that M is complete. Then f is a Riemannian covering map.*

The isometries of M obviously form a group $I(M)$. We endow $I(M)$ with the compact open topology, namely, the smallest topology for which the sets

$$W(K, U) := \{f \in I(M) \mid f(K) \subset U\},$$

where K is a compact subset of M and U is an open subset of M, are open.

Since M is a locally compact separable metric space, this topology has a countable basis ([H, p. 202]). Note that a sequence of isometries converges in the compact open topology if and only if it converges uniformly on compact subsets of M.

Theorem 2.10. [H, p. 204] *Endowed with the compact open topology, the isometry group $I(M)$ of a Riemannian manifold M is a locally compact topological transformation group of M. Moreover, for all $x \in M$, the isotropy subgroup $I(M)_x = \{g \in G \mid gx = x\}$ of $I(M)$ at x is compact.*

3. Riemannian locally symmetric spaces

Starting from the geometric definition in terms of geodesic symmetries, we prove that a Riemannian manifold is locally symmetric if and only if its Riemann curvature tensor is parallel. A good reference is [H] (see also [W]).

Definition 3.1. *Let (M,g) be a Riemannian manifold and let $m \in M$. The local geodesic symmetry s_m at m is the local diffeomorphism defined on small enough normal neighborhoods of m by $s_m = \exp_m \circ(-\mathrm{Id}_{T_mM}) \circ \exp_m^{-1}$.*

Definition 3.2. *A Riemannian manifold (M,g) is called locally symmetric if for each $m \in M$ the local geodesic symmetry at m is an isometry.*

Remark 3.3. It follows from Lemma 2.8 that a Riemannian manifold (M,g) is locally symmetric if for each $m \in M$ there exists a local isometry ϕ_m defined on a neighborhood of m such that $\phi_m(m) = m$ and whose differential $\mathrm{d}_m\phi_m$ at m is $-\mathrm{id}_{\mathrm{T_mM}}$.

Since the Levi-Civitá connection ∇ and the Riemann curvature tensor R of g are invariant by isometries, for any point m of M we have $s_m^\star(\nabla R)_m = \mathrm{d}_m s_m \circ (\nabla R)_m = -(\nabla R)_m$. But ∇R is a (4,1)-tensor and therefore $s_m^\star(\nabla R)_m = (\nabla R)_m$. Hence:

Proposition 3.4. *A Riemannian locally symmetric manifold has parallel Riemann curvature tensor : $\nabla R = 0$.*

In fact, the converse of this statement is also true, as the following more general result shows.

Theorem 3.5. *Let (M, g_M) and (N, g_N) be two Riemannian manifolds with parallel curvature tensors. Let $m \in M$ and $n \in N$. Assume that $\varphi : T_mM \longrightarrow T_nN$ is a linear isometry preserving the Riemann curvature tensors, i.e. such that for all u, v, w in T_mM, $R^N_n(\varphi(u), \varphi(v))\varphi(w) = \varphi(R^M_m(u, v)w)$. Then there exist normal neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of m and n and an isometry $f : \mathcal{U} \longrightarrow \mathcal{V}$ such that $f(m) = n$ and $\mathrm{d}_m f = \varphi$.*

Proof. Let $r > 0$ be such that $\exp_m : B(0,r) \longrightarrow \mathcal{U} = B(m,r)$ and $\exp_n : B(0,r) \longrightarrow \mathcal{V} = B(n,r)$ are diffeomorphisms, and define $f : \mathcal{U} \longrightarrow \mathcal{V}$ by $f = \exp_n \circ\varphi \circ \exp_m^{-1}$. f is a diffeomorphism. Let us prove that f is an isometry.

Let $x \in \mathcal{U}$, $x = \exp_m(v)$, and let $w \in T_xM$. Let J be the Jacobi field along the geodesic γ_v joining m to x such that $J(0) = 0$ and $J'(0) = \mathrm{d}_x(\exp_m)^{-1}(w)$. Then $J(1) = w$ by Proposition 2.4. Let $(e_1(t) = \dot\gamma_v(t), \dots, e_n(t))$ be a parallel field of orthonormal frames along the geodesic γ_v in M. In this frame, we have $J(t) = \sum_i y_i(t)e_i(t)$.

Let now $(\varepsilon_1(t), \dots, \varepsilon_n(t))$ be the parallel orthonormal frame field along the geodesic $\gamma_{\varphi(v)}$ in N starting from n such that for all i, $\varepsilon_i(0) = \varphi(e_i(0))$. Define $I(t) = \sum_i y_i(t)\varepsilon_i(t)$. Then I is a Jacobi vector field along $\gamma_{\varphi(v)}$. Indeed,

$$\begin{aligned}
g_N(I'' + R^N(\dot\gamma_{\varphi(v)}, I)\dot\gamma_{\varphi(v)}, \varepsilon_i) &= y_i'' + \textstyle\sum_j y_j\, R^N(\varepsilon_1, \varepsilon_j, \varepsilon_1, \varepsilon_i)\\
&= y_i'' + \textstyle\sum_j y_j\, R^N_n(\varepsilon_1(0), \varepsilon_j(0), \varepsilon_1(0), \varepsilon_i(0))\\
&= y_i'' + \textstyle\sum_j y_j\, R^N_n(\varphi(e_1(0)), \varphi(e_j(0)), \varphi(e_1(0)), \varphi(e_i(0)))\\
&= y_i'' + \textstyle\sum_j y_j\, R^M_m(e_1(0), e_j(0), e_1(0), e_i(0))\\
&= y_i'' + \textstyle\sum_j y_j\, R^M(e_1, e_j, e_1, e_i)\\
&= g_M(J'' + R^M(\dot\gamma_v, J)\dot\gamma_v, e_i)\\
&= 0
\end{aligned}$$

where we have used the fact that the curvature tensor R is parallel if and only if for any parallel vector fields X, Y and Z, the vector field $R(X,Y)Z$ is also parallel.

Now, $I(0) = 0$ and $I'(0) = \varphi(J'(0))$. Therefore,

$$\mathrm{d}_x f(w) = \mathrm{d}_{\varphi(v)} \exp_n(\varphi(J'(0))) = I(1)\ .$$

Since $\|I(1)\|^2_N = \sum_i |y_i(1)|^2 = \|J(1)\|^2_M$, f is an isometry. □

We therefore get:

Corollary 3.6. *A Riemannian manifold (M, g) is locally symmetric if and only if one of the following equivalent assertions is true*

(1) *the Riemann curvature tensor is parallel,*

(2) *any linear isometry from T_xM to T_yM preserving the Riemann curvature tensor (or equivalently the sectional curvatures) is induced by a (unique) local isometry between normal neighborhoods of x and y.*

Remark 3.7. It is clear from the proof of Theorem 3.5 that if γ is a geodesic through $m \in M$, then the differential at $\gamma(t)$ of the geodesic symmetry s_m is given by $\mathrm{d}_{\gamma(t)} s_m = -\gamma_t^{-t}$ where γ_t^s denotes parallel transport along γ from $T_{\gamma(t)}M$ to $T_{\gamma(s)}M$.

4. Riemannian globally symmetric spaces

Our starting point is the geometric definition of a Riemannian (globally) symmetric space M, from which we deduce some of the algebraic properties of the isometry group of M and its Lie algebra. One could also go the other way around: this is the topic of P.-E. Paradan's lecture [P]. A much more detailed exposition can be found in [H] (see also [Bo]).

4.1. Definition and first results.

Definition 4.1. *A Riemannian manifold (M, g) is said to be a Riemannian (globally) symmetric space if for all $m \in M$, the local geodesic symmetry at m extends to a global isometry of M.*

Remark 4.2. It follows from the results of the previous section that if M is locally symmetric and if $\exp_m : T_mM \longrightarrow M$ is a diffeomorphism for all m, then s_m is a global isometry and hence M is globally symmetric. This is the case for example if M is locally symmetric, simply connected, complete and non-positively curved.

Example. Let $M = P(n, \mathbb{R})$ be the open cone of positive-definite symmetric $n \times n$ matrices. The cone M is a differentiable manifold of dimension $n(n+1)/2$. The tangent space at m is isomorphic via translation to the space $S(n, \mathbb{R})$ of symmetric matrices and one can define a Riemannian metric on M by the following formula: $g_m(X, Y) = \mathrm{tr}(m^{-1}Xm^{-1}Y)$, where $m \in M$, $X, Y \in T_mM \simeq S(n, \mathbb{R})$ and $\mathrm{tr}A$ is the trace of the matrix A.

It is easily checked that the map $x \mapsto mx^{-1}m$ is an isometry of M endowed with the metric we just defined. This map fixes m and its differential at m is $-\mathrm{id}$. It is therefore the geodesic symmetry s_m at m and M is globally symmetric (cf. the remark following Definition 3.2).

Proposition 4.3. *A Riemannian globally symmetric space M is complete. Moreover, if G denotes the identity component of the isometry group of M, then G is transitive on M; namely, M is G-homogeneous.*

Proof. We can use the geodesic symmetries to extend the geodesics on $\mathbb{R}$ and hence M is complete. If now x and y are two points of M then let γ be a unit speed geodesic from x to y and consider the isometries $p_t = s_{\gamma(t/2)} \circ s_x$. Then $p_0 = \mathrm{Id}$ and hence $p_t \in G$. For $t = d(x, y)$, $p_t(x) = y$ thus G is indeed transitive on M. $\square$

Given a unit speed geodesic γ in M, the isometry $t \mapsto p_t := s_{\gamma(t/2)} \circ s_{\gamma(0)}$ of the previous proof is called a *transvection* along γ (see Lemma 4.11 below).

Let $K = G_m$ be the isotropy group at $m \in M$ of the identity component G of the isometry group of M.

We know from theorem 2.10 that endowed with the compact open topology, the group G is a locally compact topological transformation group of M and that K is a compact subgroup of G. Since G is transitive on M, this implies that the map $gK \mapsto g.m$ from G/K to M is a homeomorphism. Furthermore, one has the following result, due to Myers-Steenrod:

Theorem 4.4. [H, pp. 205-209] *The topological group G is a Lie transformation group of M and M is diffeomorphic to G/K.*

Example. The group $\mathrm{GL}^+(n,\mathbb{R})$ of invertible matrices with positive determinant acts transitively and isometrically on $M = P(n,\mathbb{R})$ by $g.m := gm\,{}^t g$. The stabilizer of $\mathrm{id} \in M$ is $\mathrm{SO}(n,\mathbb{R})$. Therefore M can be identified with $\mathrm{GL}^+(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. One should notice that if n is even, $\mathrm{GL}^+(n,\mathbb{R})$ does not act effectively on M: the identity component of the isometry group of M is $\mathrm{GL}^+(n,\mathbb{R})/\{\pm\mathrm{id}\}$.

Before analyzing in more details the structure of G and its Lie algebra, we prove that if a Riemannian manifold is locally symmetric, complete and simply connected, it is globally symmetric. In particular, this implies that the universal cover of a complete locally symmetric space is a globally symmetric space. For this, we need two lemmas [H, pp. 62-63].

Lemma 4.5. *Let M and N be complete Riemannian locally symmetric manifolds. Let $m \in M$, $\mathcal{U}$ a normal neighborhood of m and $f : \mathcal{U} \longrightarrow N$ an isometry. Let σ be a curve in M starting from m. Then f can be continued along σ, i.e. for each $t \in [0,1]$, there exists an isometry f_t from a neighborhood $\mathcal{U}_t$ of $\sigma(t)$ into N such that $\mathcal{U}_0 = \mathcal{U}$, $f_0 = f$ and there exists ε such that for $|t-s| < \varepsilon$, $\mathcal{U}_s \cap \mathcal{U}_t \neq \emptyset$ and $f_s = f_t$ on $\mathcal{U}_s \cap \mathcal{U}_t$.*

Remark 4.6. Such a continuation is unique because $f_t(\sigma(t))$ and $\mathrm{d}_{\sigma(t)} f_t$ vary continuously with t.

Proof. Assume that f is defined on a normal ball $B(x,\rho)$ around some $x \in M$ and that for some $r > \rho$, $B(x,r)$ and $B(f(x),r)$ are normal balls around x and $f(x)$. Then the map $\exp_{f(x)} \circ d_x f \circ \exp_x^{-1}$ is an isometry from $B(x,r)$ to $B(f(x),r)$ and it must coincide with f on $B(x,\rho)$ since it maps x to $f(x)$ and its differential at x equals $d_x f$. Therefore f can be extended to $B(x,r)$.

Define $I = \{t \in [0,1] \,|\, f \text{ can be extended near } \sigma(t)\}$ and $T = \sup I$. I is an open subinterval of $[0,1]$ and $0 \in I$.

Let then $q = \lim\limits_{t \to T} f_t(\sigma(t))$. This limit exists by completeness. Choose r such that $B(\sigma(T),3r)$ and $B(q,3r)$ are convex normal balls around $\sigma(T)$ and q, and let t be such that $\sigma(t) \in B(\sigma(T),r)$ and $f_t(\sigma(t)) \in B(q,r)$. Then $B(\sigma(t),2r)$ and $B(f_t(\sigma(t)),2r)$ are normal balls around $\sigma(t)$ and $f_t(\sigma(t))$. Hence f can be extended to $B(\sigma(t),2r)$, which contains $\sigma(T)$. Thus $T \in I$ and $I = [0,1]$. $\square$

Lemma 4.7. *Let M and N be complete Riemannian locally symmetric manifolds. Let $m \in M$, $\mathcal{U}$ a normal neighborhood of m and $f : \mathcal{U} \longrightarrow N$ an isometry. Let σ be a curve in M starting from m and τ be another curve, homotopic to σ with end points fixed. Call f^σ and f^τ the continuations of f along σ and τ. Then f^σ and f^τ agree in a neighborhood of $\sigma(1) = \tau(1)$.*

Proof. Let $H : [0,1]^2 \longrightarrow M$ be the homotopy between σ and τ: $\forall t,s$, $H(t,0) = \sigma(t)$, $H(t,1) = \tau(t)$, $H(0,s) = m$, $H(1,s) = \sigma(1) = \tau(1)$.

Call f^s the continuation of f along the curve $H_s : t \mapsto H(t,s)$.

Let $I = \{s \in [0,1] \,|\, \forall a \leq s,\ f^a(1) = f^0(1) = f^\sigma(1) \text{ near } \sigma(1)\}$. I is clearly an open subinterval of $[0,1]$ containing 0. Let $S = \sup I$.

The curves H_S and $f^S \circ H_S$ are continuous, hence there exists r such that for all t, $B(H_S(t),2r)$ and $B(f^S \circ H_S(t),2r)$ are normal balls. But then there exists ε such that for

$0 < S - s < \varepsilon$ and for all t, $H_s(t) \in B(H_S(t), r)$. Then f^S is a continuation of f along H_s and therefore by uniqueness $f^S = f^s$ near $\sigma(1)$. Hence $S \in I$ and $I = [0,1]$. $\square$

We may now state:

Theorem 4.8. *Let M and N be complete Riemannian locally symmetric spaces. Assume that M is simply connected. If $m \in M$, $n \in N$, and $\varphi : T_mM \longrightarrow T_nN$ is a linear isometry preserving the Riemann curvature tensors, then there exists a unique Riemannian covering $f : M \longrightarrow N$ such that $f(m) = n$ and $\mathrm{d}_m f = \varphi$.*

Proof. It follows from the lemmas above that setting $f(\exp_m(v)) = \exp_n(\varphi(v))$ gives a well-defined map f from M onto N. Moreover this map is a local isometry and since M is complete, it is a Riemannian covering map by Proposition 2.9. $\square$

Corollary 4.9. *Let M be a complete simply connected Riemannian manifold. The following conditions are equivalent:*
(1) *M is locally symmetric,*
(2) *M is globally symmetric,*
(3) *Any linear isometry between T_xM and T_yM preserving the Riemann curvature tensor (or equivalently the sectional curvatures) is induced by an (unique) isometry of M.*

4.2. **Structure of the Lie algebra of the isometry group.**

Let now (M, g) be a globally symmetric Riemannian space, G the identity component of the isometry group of M, $m \in M$, $s = s_m$ the geodesic symmetry at m, and K the isotropy group of m in G.

The isotropy group K is a compact subgroup of G and it follows from what we have seen that the linear isotropy representation $k \in K \mapsto \mathrm{d}_m k$ identifies K with the (closed) subgroup of $\mathrm{O}(T_mM, g_m)$ consisting of linear isometries which preserve the curvature tensor R_m.

Recall that M is identified with the quotient G/K. We call m the map $G \longrightarrow M$, $g \mapsto g.m$.

The Lie algebra $\mathfrak{g}$ of G can be seen as a Lie algebra of Killing vector fields on M: if $X \in \mathfrak{g}$, the corresponding vector field $X^\star$ is defined by $X^\star(m) = \frac{\mathrm{d}}{\mathrm{d}t} e^{tX} m|_{t=0}$, for any $m \in M$. Then, $X^\star(m) = \mathrm{d}_e\omega_m$ where ω_m denotes the orbit map $G \to M$ defined by $g \mapsto g.m$. It should also be noted that, under this identification, $[X, Y]^\star = -[X^\star, Y^\star]$, where in the right-hand side, $[\,,\,]$ denotes the usual bracket of vector fields on M.

The symmetry s induces an involution σ of G given by $\sigma(g) = sgs$ and the differential $\mathrm{d}_e\sigma = \mathrm{Ad}(s)$ is an involution of the Lie algebra $\mathfrak{g}$ of G.

We therefore have a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the $+1$ and -1 eigenspaces of $\mathrm{Ad}(s)$. Note that since $\mathrm{Ad}(s)[X, Y] = [\mathrm{Ad}(s)X, \mathrm{Ad}(s)(Y)]$ for all $X, Y \in \mathfrak{g}$, we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, i.e. $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, i.e. $\mathfrak{p}$ is $\mathrm{ad}(\mathfrak{k})$-invariant, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Such a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ associated to m.

Proposition 4.10. *The group K lies between $G^\sigma := \{g \in G \,|\, \sigma g = g\}$ and G_0^σ, the identity component of G^σ. The Lie algebra $\mathfrak{k}$ of K is also the kernel of $\mathrm{d}_e m : \mathfrak{g} \longrightarrow T_mM$. Consequently, $\mathrm{d}_e m_{|\mathfrak{p}} : \mathfrak{p} \longrightarrow T_mM$ is an isomorphism.*

Proof. Let $k \in K$. Then $sks(m) = m = k(m)$ and $\mathrm{d}_m(sks) = -\mathrm{Id} \circ \mathrm{d}_m k \circ (-\mathrm{Id}) = \mathrm{d}_m k$, hence $sks = k$. Thus $K \subset G^\sigma$ and $\mathrm{Lie}(K) \subset \mathfrak{k}$.

Now, let $X \in \mathfrak{k}$. This is equivalent to $e^{tX} \in G_0^\sigma$ since $se^{tX}s = e^{t\mathrm{Ad}(s)X} = e^{tX}$. Then $e^{tX}m$ is fixed by s for all t. Since m is an isolated fixed point of s, we have $e^{tX}m = m$ for all t. Thus $G_0^\sigma \subset K$ and $\mathfrak{k} \subset \mathrm{Lie}(K)$.

If $X \in \mathfrak{k}$, then $\mathrm{d}_e m(X) = \frac{\mathrm{d}}{\mathrm{d}t} e^{tX} m|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} m|_{t=0} = 0$. On the other hand, assume that $X \in \mathfrak{g}$ is such that $\mathrm{d}_e m(X) = 0$. Let $f : M \longrightarrow \mathbb{R}$ be any function and let h be the function on M defined by $h(p) = f(e^{aX}p)$ for some $a \in \mathbb{R}$. Then

$$0 = \mathrm{d}_m h(\mathrm{d}_e m X) = \frac{\mathrm{d}}{\mathrm{d}t} h(e^{tX}m)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(e^{aX}e^{tX}m)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(e^{tX}m)|_{t=a} .$$

Hence $t \mapsto f(e^{tX}m)$ is constant. This implies $e^{tX} \in K$ and $X \in \mathfrak{k}$.

The map $\mathrm{d}_e m_{|\mathfrak{p}}$ is therefore injective. Since $\mathfrak{p}$ and T_mM have the same dimension, we are done. $\square$

Finally, the scalar product g_m on T_mM gives a positive definite inner product Q on $\mathfrak{p}$ which is $\mathrm{ad}(\mathfrak{k})$-invariant. Indeed, for $X \in \mathfrak{k}$ and $V, W \in \mathfrak{p}$, $Q([X,V],W) + Q(V,[X,W]) = g_m([X,V]^\star(m), W^\star(m)) + g_m(V^\star(m), [X,W]^\star(m)) = (X.g(V,W))|_m = 0$ since $X^\star(m) = 0$. This inner product can be extended to $\mathfrak{g}$ by choosing any $\mathrm{ad}(\mathfrak{k})$-invariant inner product on $\mathfrak{k}$.

Altogether, these data define what is called a structure of *orthogonal involutive Lie algebra* on $\mathfrak{g}$.

Example. In the case of the symmetric space $M = P(n,\mathbb{R}) = \mathrm{GL}^+(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, the involution σ of $\mathrm{GL}^+(n,\mathbb{R})$ corresponding to the geodesic symmetry $s = s_{\mathrm{id}} : x \mapsto x^{-1}$ is easily seen to be the map $g \mapsto {}^t g^{-1}$. Its differential at e is the map $X \mapsto -{}^tX$. Therefore the Cartan decomposition of $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$ is just the decomposition of a matrix into its symmetric and skew-symmetric parts: $\mathfrak{k} = \mathfrak{so}(n,\mathbb{R})$ and $\mathfrak{p} = S(n,\mathbb{R})$.

We end this section with a little lemma about transvections along a geodesic.

Lemma 4.11. *Let $v \in T_mM$ and let $\gamma : t \mapsto \exp_m(tv)$ be the corresponding geodesic. The transvections $p_t = s_{\gamma(t/2)}s_m$ along γ form a 1-parameter group of isometry. Moreover, if $X \in \mathfrak{p}$ is such that $\mathrm{d}_e m(X) = v$, then $p_t = e^{tX}$, so that in particular $e^{tX}m = \gamma(t)$ and $\mathrm{d}_m e^{tX} = \gamma_0^t$, the parallel transport along γ from T_mM to $T_{\gamma(t)}M$.*

Proof. Clearly, $p_t(\gamma(u)) = \gamma(u+t)$. Moreover, $\mathrm{d}_{\gamma(u)}p_t : T_{\gamma(u)}M \longrightarrow T_{\gamma(u+t)}M$ is parallel transport along γ. Indeed $\mathrm{d}_{\gamma(u)}p_t = \mathrm{d}_{\gamma(u)}(s_{\gamma(t/2)}s_m) = d_{\gamma(-u)}s_{\gamma(t/2)} \circ d_{\gamma(u)}s_m = \gamma_{-u}^{u+t} \circ \gamma_u^{-u} = \gamma_u^{u+t}$. Therefore, $p_tp_u = p_{u+t}$ since they agree at m along with their differentials. $t \mapsto p_t$ is hence a 1-parameter group of isometries. Thus there exists $X \in \mathfrak{g}$ such that $p_t = e^{tX}$. Now, $\mathrm{d}_e m(X) = \frac{\mathrm{d}}{\mathrm{d}t} p_t m|_{t=0} = v$. $\square$

4.3. **Further identifications and curvature computation.**

As we said, $\mathfrak{p}$ can be identified with T_mM, whereas $\mathfrak{k}$ can be identified with a subalgebra $\mathfrak{t}$ of $\mathfrak{o}(T_mM, g_m)$. More precisely,

$$\mathfrak{t} = \{T \in \mathfrak{o}(T_mM, g_m) \,|\, \forall u, v \in T_mM,\ T \circ R_m(u,v) = R_m(Tu,v) + R_m(u,Tv) + R_m(u,v) \circ T\} .$$

We will denote by T_X the element of $\mathfrak{t}$ corresponding to $X \in \mathfrak{k}$.

Therefore, $\mathfrak{g}$ is isomorphic to $\mathfrak{t} \oplus T_mM$ as a vector space. We will now see what is the Lie algebra structure induced on $\mathfrak{t} \oplus T_mM$ by this isomorphism.

Let $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, and let f be a function on M. Then,

$$[X,Y]^\star.f = -[X^\star, Y^\star].f = Y^\star.X^\star.f - X^\star.Y^\star.f .$$

But $(X^\star.(Y^\star.f))(m) = 0$ since $X^\star(m) = 0$. On the other hand, $X^\star.f = \lim_{t\longrightarrow 0} \frac{1}{t}(f \circ e^{tX} - f)$. Therefore,

$$\begin{array}{rcl}
(Y^\star.(X^\star.f))(m) & = & \lim_{t\longrightarrow 0} \frac{1}{t}\big(Y^\star.(f\circ e^{tX})(m) - (Y^\star.f)(m)\big) \\
& = & \lim_{t\longrightarrow 0} \frac{1}{t}\big(\mathrm{d}_m f \circ \mathrm{d}_m e^{tX}(Y^\star(m)) - \mathrm{d}_m f(Y^\star(m))\big) \\
& = & \mathrm{d}_m f\Big(\lim_{t\longrightarrow 0} \frac{1}{t}(\mathrm{d}_m e^{tX}(Y^\star(m)) - Y^\star(m))\Big) \\
& = & \mathrm{d}_m f\big(T_X(Y^\star(m))\big)
\end{array}$$

Hence $[X,Y]^\star(m) = T_X(Y^\star(m))$.

Let now $X, Y, Z \in \mathfrak{p}$ and $X^\star$, $Y^\star$, $Z^\star$ the corresponding Killing fields on M. First, it is immediate from the formula for the Levi-Civitá connection that

$$2g(\nabla_{X^\star}Y^\star, Z^\star) = g([X^\star, Y^\star], Z^\star) + g([Y^\star, Z^\star], X^\star) + g([X^\star, Z^\star], Y^\star) \ ,$$

since $X^\star.g(Y^\star, Z^\star) = g([X^\star, Y^\star], Z^\star) + g(Y^\star, [X^\star, Z^\star])$. We hence have $(\nabla_{X^\star}Y^\star)(m) = 0$. The Riemann curvature tensor is given by $R(X^\star, Y^\star)Z^\star = \nabla_{[X^\star,Y^\star]}Z^\star - \nabla_{X^\star}\nabla_{Y^\star}Z^\star + \nabla_{Y^\star}\nabla_{X^\star}Z^\star$. Hence, dropping the upper-script * for the computation,

$$\begin{array}{rcl}
R(X,Y,X,Y) & = & g(\nabla_{[X,Y]}X, Y) - g(\nabla_X\nabla_Y X, Y) + g(\nabla_Y\nabla_X X, Y) \\
& = & -g(\nabla_Y X, [X,Y]) - X.g(\nabla_Y X, Y) + g(\nabla_Y X, \nabla_X Y) \\
& & \qquad\qquad\qquad + Y.g(\nabla_X X, Y) - g(\nabla_X X, \nabla_Y Y) \\
& = & \|\nabla_Y X\|^2 + Y.g([X,Y], X) - g(\nabla_X X, \nabla_Y Y) \\
& = & \|\nabla_Y X\|^2 + g([Y,[X,Y]], X) - \|[X,Y]\|^2 - g(\nabla_X X, \nabla_Y Y)
\end{array}$$

Therefore, at m, $R_m(X^\star(m), Y^\star(m), X^\star(m), Y^\star(m)) = g_m([[X^\star, Y^\star], X^\star](m), Y^\star(m))$. This implies that the curvature tensor is given by

$$R_m(X^\star(m), Y^\star(m))Z^\star(m) = [[X^\star, Y^\star], Z^\star](m) = [[X,Y],Z]^\star(m) = T_{[X,Y]}(Z^\star(m)) \ .$$

Thus $T_{[X,Y]} = R_m(X^\star(m), Y^\star(m))$.

One then checks easily that if $X, Y \in \mathfrak{k}$, $T_{[X,Y]} = T_X T_Y - T_Y T_X$.

Summarizing, we have

Proposition 4.12. *The Lie algebra structure on $\mathfrak{g} = \mathfrak{t} \oplus T_m M$ is given by:*
$[T,S] = TS - ST$ *for* $T, S \in \mathfrak{t}$*;*
$[T,u] = -[u,T] = T(u)$ *for* $T \in \mathfrak{t}$ *and* $u \in T_m M$*;*
$[u,v] = R_m(u,v)$ *for* $u, v \in T_m M$*.*

Remark 4.13. For any Riemannian *locally* symmetric space (M, g), the Lie algebra $\mathfrak{t} \oplus T_m M$ is defined and is an orthogonal involutive Lie algebra. It is the infinitesimal version of the isometry group of a globally symmetric space.

5. Riemannian manifolds of non-positive curvature

In this section we review some of the most important "comparison" results for manifolds of non-positive curvature. They will be useful in our study of symmetric space of non-compact type. We will stick to Riemannian manifolds but most of these results generalize to the setting of metric spaces (see the remark following Proposition 5.11). Good references for the material in this section are the books [Ba] and [BH] (and also [E]).

5.1. **The Rauch comparison theorem.**

Before specializing to non-positive curvature, we prove (see also [dC, chap. 10]) the following

Theorem 5.1 (Rauch comparison theorem)**.** *Let M be a Riemannian manifold and let $\gamma : [0,T) \longrightarrow M$ be a unit speed geodesic. Assume that all the sectional curvatures of M along γ are bounded from above by some real number κ. Let Y be a normal Jacobi field along γ. Then, for all t such that $\|Y\|(t) \neq 0$, we have*

$$\|Y\|''(t) + \kappa\|Y\|(t) \geq 0 \ .$$

In particular, if y_κ is the solution of the differential equation $y'' + \kappa y = 0$, with the same initial conditions as $\|Y\|$, then $\|Y\|(t) \geq y_\kappa(t)$ for $t \in [0,T)$.

Proof. This is just a computation. $\|Y\|' = \langle Y, Y'\rangle \|Y\|^{-1}$, hence

$$\begin{aligned} \|Y\|'' &= (\langle Y, Y''\rangle + \langle Y', Y'\rangle)\|Y\|^{-1} - \langle Y, Y'\rangle^2\|Y\|^{-3} \\ &= \|Y'\|^2\|Y\|^{-1} - \langle R(\dot\gamma, Y)\dot\gamma, Y\rangle\|Y\|^{-1} - \langle Y, Y'\rangle^2\|Y\|^{-3} \\ &\geq \|Y'\|^2\|Y\|^{-1} - \kappa\|Y\| - \langle Y, Y'\rangle^2\|Y\|^{-3} \end{aligned}$$

where for the second equality we used the definition of a Jacobi field, and for the inequality the fact that Y is normal to γ. Thus

$$\|Y\|'' + \kappa\|Y\| \geq \|Y\|^{-3}(\|Y'\|^2\|Y\|^2 - \langle Y, Y'\rangle^2) \geq 0$$

by Cauchy-Schwarz inequality.

Let now $f := \|Y\|'y_\kappa - \|Y\|y_\kappa'$. Then $f(0) = 0$ and $f' = \|Y\|''y_\kappa - \|Y\|y_\kappa'' \geq -\|Y\|(y_\kappa'' + \kappa y_\kappa) = 0$. Hence $f \geq 0$ and therefore $(\|Y\|/y_\kappa)' \geq 0$ and we are done. $\square$

This result allows to compare different geometric quantities in a manifold M all of whose sectional curvatures are bounded from above by κ to corresponding quantities in a simply connected manifold M_κ of constant sectional curvature κ. Recall that M_κ is unique up to isometry. By scaling the metric, we can assume $\kappa \in \{-1, 0, 1\}$, and the corresponding model spaces of dimension n are hyperbolic n-space $M_{-1} = \mathbb{H}^n$, Euclidean n-space $M_0 = \mathbb{E}^n$ and the n-sphere $M_1 = \mathbb{S}^n$ with its standart metric.

Corollary 5.2. *Let M be a Riemannian manifold all of whose sectional curvatures are bounded from above by $\kappa \in \mathbb{R}$. Let M_κ be the model space of constant sectional curvature κ (of the same dimension as M). Let $m \in M$, $p \in M_\kappa$ and φ a linear isometry between T_mM and T_pM_κ. Let r be so small that $B(m,r)$ and $B_\kappa(p,r)$ are normal convex neighborhoods of m in M and p in M_κ. Let $f : B(m,r) \longrightarrow B_\kappa(p,r)$ be given by $f = \exp_p \circ\varphi \circ \exp_m^{-1}$. Then f is distance non-increasing.*

Proof. Let $x \in B(m,r)$, $x = \exp_m(v)$, and γ the geodesic $t \mapsto \exp_m(tv)$. Let $w \in T_xM$, and call $w^\perp$ the component of w orthogonal to $\dot\gamma(1)$ and $w^T = w - w^\perp$.

Let Y be the Jacobi field along γ such that $Y(0) = 0$ and $Y(1) = w$. We can also write $Y = Y^T + Y^\perp$ where $Y^T(t) = \frac{\|w^T\|}{\|\dot\gamma(1)\|}t\dot\gamma(t)$ is a Jacobi field along γ collinear to $\dot\gamma$ such that $Y^T(1) = w^T$ and $Y^\perp = Y - Y^T$ is a normal Jacobi vector field along γ such that $Y^\perp(1) = w^\perp$.

Call Y_κ the Jacobi field along the geodesic $\exp_p(t\varphi(v))$ in M_κ such that $Y_\kappa(0) = 0$ and $Y_\kappa'(0) = \varphi(Y'(0))$. With the obvious notation, we have $Y_\kappa = Y_\kappa^T + Y_\kappa^\perp$.

Then $\mathrm{d}_xf(w) = Y_\kappa(1)$. Hence

$$\|\mathrm{d}_xf(w)\|^2 = \|Y_\kappa(1)\|^2 = \|Y_\kappa^T(1)\|^2 + \|Y_\kappa^\perp(1)\|^2 \ .$$

Now, $\|Y_\kappa^T(1)\| = \|Y^T(1)\|$ and it follows from the Rauch comparison theorem that $\|Y^\perp\| \geq y_\kappa = \|Y_\kappa^\perp\|$. Hence $\|\mathrm{d}_xf(w)\| \leq \|w\|$.

Therefore if x and y are two points in $B(m,r)$ and if $\gamma \subset B(m,r)$ is the geodesic joining these two points, then $d_\kappa(f(x), f(y)) \leq \mathrm{L}(f \circ \gamma) \leq \mathrm{L}(\gamma) = d(x,y)$. $\square$

5.2. **Hadamard manifolds.**

From now on, we will focus on the case $\kappa = 0$, namely, (M,g) is non-positively curved.

Definition 5.3. *A complete simply connected non-positively curved manifold is called a Hadamard manifold.*

It follows immediately from the Rauch comparison theorem that in a Hadamard manifold M, a Jacobi vector field Y along a geodesic γ such that $Y(0) = 0$ never vanishes again. This implies that for all $m \in M$, $\exp_m$ is a local diffeomorphism from T_mM onto M (since M is complete). Endowing T_mM with the metric $\exp_m^\star g$, $\exp_m$ becomes a local isometry. Now, $(T_mM, \exp_m^\star g)$ is complete since the geodesics through 0 are straight lines. Hence $\exp_m$ is a covering map and since M is simply connected, $\exp_m$ is a diffeomorphism:

Theorem 5.4. *A Hadamard space of dimension n is diffeomorphic to $\mathbb{R}^n$.*

Note that two points in a Hadamard manifold are joined by a unique minimizing geodesic.

Until the end of this section, M will be a Hadamard manifold and $\mathbb{E}$ will be Euclidean 2-space. We will assume all geodesics parametrized by arc length.

5.2.1. *Geodesic triangles in Hadamard manifolds. The CAT(0) Property.*

Given three points p, q, r in M (or in $\mathbb{E}$) we will denote by $\sphericalangle_p(q,r)$ the angle between the geodesic segments $[p,q]$ and $[p,r]$ emanating from p, that is, the Riemannian angle between the tangent vectors to these geodesics at p.

Definition 5.5. *A geodesic triangle T in a Riemannian manifold consists of three points p, q, r, its vertices, and three geodesic arcs $[p,q]$, $[q,r]$ and $[r,p]$ joining them, its sides or edges. Note that in a Hadamard manifold a geodesic triangle is determined by its vertices.*

We will sometimes denote by $\hat{p}$ (resp. $\hat{q}$, $\hat{r}$) the vertex angle of a geodesic triangle $T = T(p,q,r)$ at p (resp. q, r), i.e. $\hat{p} = \sphericalangle_p(q,r)$.

Definition 5.6. *A comparison triangle of a geodesic triangle $T \subset M$ in $\mathbb{E}$ is a geodesic triangle T_0 in $\mathbb{E}$ whose side lengths equal the side lengths of T. Such a triangle always exists and is unique up to isometries of $\mathbb{E}$.*

Given an "object" a in a geodesic triangle T in M, we will always denote by a_0 the comparison object in the comparison triangle T_0. For example, if p is a vertex of T, p_0 will be the corresponding vertex of T_0. If x is a point on the side $[p,q]$ of T, x_0 will be the point on the comparison side $[p_0,q_0]$ of T_0 such that $d_0(p_0,x_0) = d(p,x)$.

We begin with the following remark concerning angles.

Lemma 5.7. [A] *The Riemannian angle between two unit tangent vectors $u, v \in T_mM$ is the limit as t goes to zero of the vertex angle at m_0 of the comparison triangle of $T(m, \sigma_u(t), \sigma_v(t))$.*

Proof. It follows from Corollary 5.2 that $d(\sigma_u(t), \sigma_v(t)) \geq t\|u - v\|$. Now, consider the path $c : s \mapsto \exp_m(tu + st(v-u))$ from $\sigma_u(t)$ to $\sigma_v(t)$.

$$d(\sigma_u(t), \sigma_v(t)) \leq L(c) = t \int_0^1 \|\mathrm{d}_{t(u+s(v-u))} \exp_m(v-u)\| ds \ .$$

For t close to 0, $\mathrm{d}_{t(u+s(v-u))}\exp_m$ is close to Id and therefore we see that

$$\lim_{t\longrightarrow 0}\frac{d(\sigma_u(t),\sigma_v(t))}{t\|u-v\|}=1\ .$$

This implies that the triangle $T(0,tu,tv)$ in T_mM goes to the comparison triangle of $T(m,\sigma_u(t),\sigma_v(t))$ as $t\longrightarrow 0$, hence the result. $\square$

Let us start to compare geodesic triangles in a Hadamard manifolds with triangles in Euclidean space.

Lemma 5.8. *Let $m\in M$ and $u,v\in T_mM$. Let σ_u and σ_v be the corresponding unit speed geodesics. Let $x=\sigma_u(s)$ and $y=\sigma_v(t)$. Let also m_0, x_0 y_0 be points in $\mathbb{E}$ such that $d_0(m_0,x_0)=s$, $d_0(m_0,y_0)=t$ and the angle $\sphericalangle_{m_0}(x_0,y_0)$ equals the angle between u and v (see Figure 1). Then $d_0(x_0,y_0)\le d(x,y)$.*

Consequently, if α, β, γ are the vertex angles of a geodesic triangle T in M and α_0, β_0, γ_0 the corresponding vertex angles of its comparison triangle T_0, then

$$\alpha\le\alpha_0,\ \ \beta\le\beta_0,\ \ \text{and }\gamma\le\gamma_0\ .$$

In particular, $\alpha+\beta+\gamma\le\pi$.

Proof. Immediate from Corollary 5.2. $\square$

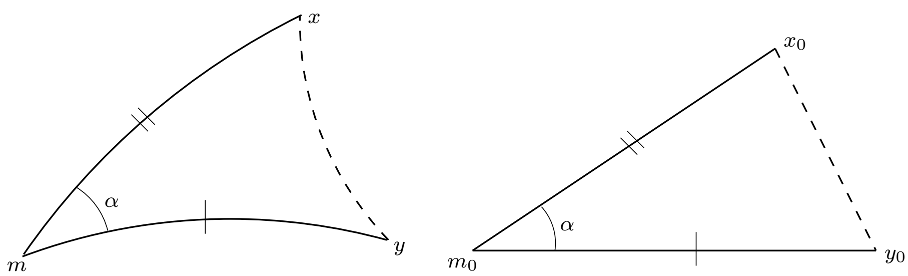

FIGURE 1.

Lemma 5.9. *Let $T=T(p,q,r)$ be a geodesic triangle in M and let T_0 be its comparison triangle in $\mathbb{E}$. Let x be a point on the side $[q,r]$. Then $d(p,x)\le d_0(p_0,x_0)$. Moreover, if the sum of the vertex angles of T equals π then $d(p,x)=d_0(p_0,x_0)$.*

Proof. Consider the geodesic triangles $T'=T(p,q,x)$ and $T''=T(p,x,r)$ and call T_0' and T_0'' their respective comparison triangles in $\mathbb{E}$. We can assume that T_0' and T_0'' are such that $p_0'=p_0''$ and $x_0'=x_0''$, and that they lie on different sides of the line through p_0' and x_0' (see Figure 2).

If $\hat{x}'$, resp. $\hat{x}''$, is the vertex angle at x of T', resp. T'', and if $\hat{x}_0'$, resp. $\hat{x}_0''$, is the corresponding vertex angle in T_0', resp. T_0'', then $\hat{x}_0'+\hat{x}_0''\ge\hat{x}'+\hat{x}''=\pi$. This implies that if we want to straighten the union $T_0'\cup T_0''$ to form a comparison triangle for T (without modifying the side lengths of T_0' and T_0'' other that $[p_0',x_0']$), we have to increase (at least not decrease) the distance from p_0 to x_0. Hence the first part of the result.

Now assume that the sum of the vertex angles of T is π. Call $\hat{p}'$ and $\hat{q}'$, resp. $\hat{p}''$ and $\hat{r}''$, the remaining vertex angles of T', resp. T''. Then $\hat{p}'+\hat{p}''+\hat{q}'+\hat{x}'+\hat{x}''+\hat{r}''=2\pi$. Since $\hat{p}'+\hat{q}'+\hat{x}'\le\pi$ and $\hat{p}''+\hat{x}''+\hat{r}''\le\pi$, we have in fact $\hat{p}'+\hat{q}'+\hat{x}'=\pi$ and $\hat{p}''+\hat{x}''+\hat{r}''=\pi$, hence all these vertex angles are equal to their comparison angles. This implies that $\hat{x}_0'+\hat{x}_0''=\hat{x}'+\hat{x}''=\pi$, hence that $d(p,x)=d_0(p_0,x_0)$. $\square$

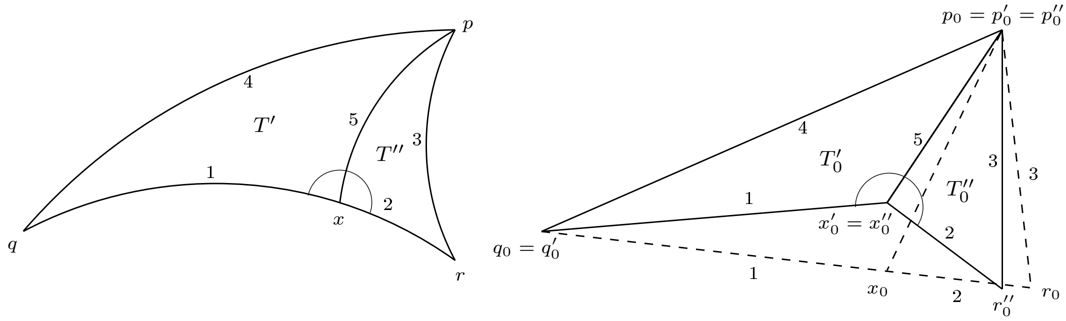

FIGURE 2.

We can now state the main property of geodesic triangles in Hadamard manifolds.

Definition 5.10. *A geodesic triangle T in a manifold is said to be CAT(0) if it is thinner than its comparison triangle in $\mathbb{E}$, namely if for any two points x and y on T, and for x_0, y_0 the corresponding points in the comparison triangle T_0 of T in $\mathbb{E}$, we have $d(x, y) \leq d_0(x_0, y_0)$.*

Proposition 5.11. (1) *Geodesic triangles in a Hadamard manifold M are CAT(0).*

(2) *Moreover, if the sum of the vertex angles of a geodesic triangle T of M equals π, then there exists a unique isometry Φ from the convex hull* $\mathrm{Conv}(T_0)$ *of T_0 in $\mathbb{E}$ into the convex hull* $\mathrm{Conv}(T)$ *of T in M, such that $\Phi(x_0) = x$ for all $x_0 \in T_0$, that is to say, T bounds a flat solid triangle in M.*

Proof. Let us first prove (1). Let x and y be two points in the triangle $T = T(p, q, r)$, and $T_0 = T(p_0, q_0, r_0)$ the comparison triangle of T. We can assume x and y are not on the same side of T, say $x \in [q, r]$ and $y \in [p, q]$. We know from Lemma 5.9 that $d(p, x) \leq d_0(p_0, x_0)$. Consider the comparison triangle $T_0' = T(p_0', x_0', q_0')$ of $T(p, x, q)$. Then, again from Lemma 5.9, $d(x, y) \leq d_0(x_0', y_0')$. Now, the lengths of the sides $[p_0', q_0']$ and $[x_0', q_0']$ of T_0' are equal to those of $[p_0, q_0]$ and $[x_0, q_0]$ in T_0, whereas $[p_0', x_0']$ is shorter than $[x_0, p_0]$. This implies that $[x_0', y_0']$ is shorter than $[x_0, y_0]$, hence that $d(x, y) \leq d(x_0, y_0)$.

Proof of (2). The assumption is $\hat{p} = \hat{p}_0$, $\hat{q} = \hat{q}_0$ and $\hat{r} = \hat{r}_0$. From the second assertion in Lemma 5.9 and from the proof of part (1) we get that $d(x, y) = d(x_0, y_0)$ for all $x, y \in T$. Now we want to define Φ in the interior of $\mathrm{Conv}(T_0)$. Let y_0 be a point there and call z_0 the unique point on the side $[q_0, r_0]$ such that $y_0 \in [p_0, z_0]$. Map z_0 to its corresponding point z on the side $[q, r]$ of T. It follows from what we just seen that the triangle $T(p_0, q_0, z_0)$ is the comparison triangle of $T(p, q, z)$. Since again the vertex angles are the same, the comparison map between these triangles is an isometry and we can map $z_0 \in [p_0, z_0]$ to the corresponding point $\Phi(z_0) \in [p, z]$. One then checks easily that Φ is isometric. $\square$

Remark 5.12. Property (1) gives one way to generalize the notion of non-positive curvature to metric spaces. Namely, we say that an (interior) metric space (X, d) has non positive curvature in the sense of Alexandrov if every point in X has an open neighborhood $\mathcal{U}$ such that any two points in $\mathcal{U}$ can be joined by a minimizing geodesic and every geodesic triangle in $\mathcal{U}$ is CAT(0). (X, d) is called a CAT(0)-space if every geodesic triangle in X is CAT(0). A complete simply connected interior metric space of non positive curvature is called a Hadamard space.

It should also be noted that, as was proved by Alexandrov in [A], a smooth Riemannian manifold has non positive curvature in the sense of Alexandrov if and only if all its sectional curvatures are non positive (see [BH, p.173] for a proof using Proposition 2.6).

Corollary 5.13 (Flat quadrilateral theorem)**.** *Let p, q, r, s be four points in M and let $\alpha = \sphericalangle_p(q,s)$, $\beta = \sphericalangle_q(p,r)$, $\gamma = \sphericalangle_r(q,s)$, $\delta = \sphericalangle_r(p,r)$. Then if $\alpha+\beta+\gamma+\delta \geq 2\pi$, this sum equals 2π and p, q, r, s "bound" a convex region in M isometric to a convex quadrilateral in $\mathbb{E}$.*

Proof. Let $T = T(p,q,s)$ and $T' = T(q,r,s)$. Call $\hat{p}$, $\hat{q}$, $\hat{s}$ and $\hat{q}'$, $\hat{r}'$, $\hat{s}'$ the vertex angles of T and T'. It follows from the triangle inequality that $\beta \leq \hat{q}+\hat{q}'$ and $\delta \leq \hat{s}+\hat{s}'$. Hence, if $\alpha+\beta+\gamma+\delta \geq 2\pi$, then $\hat{p}+\hat{q}+\hat{s} \geq \pi$ and $\hat{q}'+\hat{r}'+\hat{s}' \geq \pi$. Therefore all these inequalities are in fact equalities and the triangles T and T' are flat. Let $T_0 = T(p_0,q_0,s_0)$ and $T_0' = T(q_0,r_0,s_0)$ be comparison triangles for T and T' so that p_0 and r_0 lie on opposite sides of the line through q_0 and s_0. Then the quadrilateral $Q_0 = (p_0,q_0,r_0,s_0)$ is convex. Let $x_0 \in \mathrm{Conv}(T_0)$ and $x_0' \in \mathrm{Conv}(T_0')$. The fact that $\hat{q}+\hat{q}' = \beta$ implies that $\sphericalangle_q(x,x') = \sphericalangle_{q_0}(x_0,x_0')$, where x, resp. x', is the image of x_0, resp. x_0', under the isometry $\mathrm{Conv}(T_0) \longrightarrow \mathrm{Conv}(T)$, resp. $\mathrm{Conv}(T_0') \longrightarrow \mathrm{Conv}(T')$. This shows that these isometries patch together to give an isometry between $\mathrm{Conv}(p_0,q_0,r_0,s_0)$ and $\mathrm{Conv}(p,q,r,s)$. $\square$

5.2.2. *Convexity properties of Hadamard manifolds. Parallel geodesics.*

A Hadamard manifold shares many convexity properties with Euclidean space. Recall that a function $f : M \longrightarrow \mathbb{R}$ is convex if its restriction to each geodesic σ of M is convex.

Lemma 5.9 immediately implies

Lemma 5.14. *Let $m \in M$. The function $x \mapsto d(x,m)$ is convex.*

We also have

Proposition 5.15. *Let σ and τ be two (unit speed) geodesics in M. The function $t \mapsto d(\sigma(t),\tau(t))$ is convex.*

Proof. Let $t_1 < t_2$ and let $t = \frac{1}{2}(t_1+t_2)$. Call γ the geodesic segment from $\sigma(t_1)$ to $\tau(t_2)$ (see Figure 3).

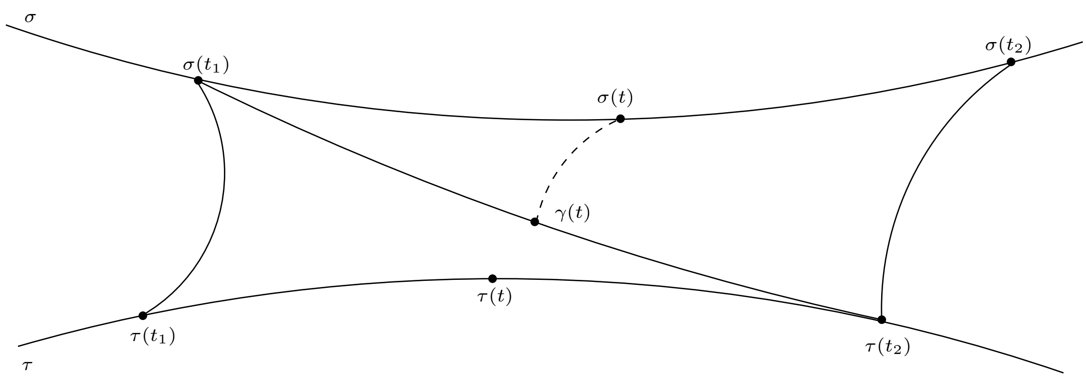

FIGURE 3.

We have $d(\sigma(t),\tau(t)) \leq d(\sigma(t),\gamma(t)) + d(\gamma(t),\tau(t))$. From the CAT(0) property, $d(\sigma(t),\gamma(t)) \leq \frac{1}{2}d(\sigma(t_2),\tau(t_2))$ since equality holds in the comparison triangle of $T(\sigma(t_1),\sigma(t_2),\tau(t_2))$. In the same way, $d(\gamma(t),\tau(t)) \leq \frac{1}{2}d(\sigma(t_1),\tau(t_1))$. This implies the proposition. $\square$

More generally, the following proposition holds:

Proposition 5.16. *Let $C \subset M$ be a closed convex set. Then for every $x \in M$ there exists a unique point $\pi_C(x) \in C$ such that $d(x, \pi_C(x)) = d(x, C)$. Moreover the map $\pi_C : x \mapsto \pi_C(x)$ is 1-Lipschitz and the function $x \mapsto d(x, C)$ is convex.*

Definition 5.17. *Two (unit speed) geodesics σ_1 and σ_2 in M are called parallel if there exists $k > 0$ such that $\forall t \in \mathbb{R}$, $d(\sigma_1(t), \sigma_2) \leq k$ and $d(\sigma_2(t), \sigma_1) \leq k$.*

Corollary 5.18 (Flat strip theorem)**.** *Let σ_1 and σ_2 be two parallel geodesics in M. Then σ_1 and σ_2 bound a flat strip, namely, there exist $D \in \mathbb{R}$ and an isometry Φ from $\mathbb{R} \times [0, D]$ with its Euclidean metric into M such that (up to affine reparametrizations of σ_1 and σ_2), $\Phi(t, 0) = \sigma_1(t)$ and $\Phi(t, D) = \sigma_2(t)$, $\forall t \in \mathbb{R}$.*

Proof. The function $t \mapsto d(\sigma_1(t), \sigma_2(t))$ is convex and bounded on $\mathbb{R}$, hence constant, say equal to $D \in \mathbb{R}$. We can assume that the closest point to $p := \sigma_1(0)$ on $\sigma_2(\mathbb{R})$ is $q := \sigma_2(0)$. We claim that for $t \neq 0$, the angle $\sphericalangle_q(p, \sigma_2(t)) \geq \frac{\pi}{2}$. If not, then by Lemma 5.7 there is a point x in the geodesic segment $[q, p]$ and a point y on the geodesic segment $[q, \sigma_2(t)]$ such that the vertex angle at q_0 of the comparison triangle $T(q_0, x_0, y_0)$ of $T(q, x, y)$ is strictly less than $\frac{\pi}{2}$. This would implies that there are points x' on $[q, x]$ and y' on $[q, y]$ such that $d(x', y') < d(x', q)$. But then q wouldn't be the point on $\sigma_2(\mathbb{R})$ closest to p. Hence, for all $t \neq 0$, $\sphericalangle_{\sigma_2(0)}(\sigma_1(0), \sigma_2(t)) = \frac{\pi}{2}$, and $p = \sigma_1(0)$ is the point on $\sigma_1(\mathbb{R})$ closest to $q = \sigma_2(0)$ so that for all $t \neq 0$ we also have $\sphericalangle_{\sigma_1(0)}(\sigma_2(0), \sigma_1(t)) = \frac{\pi}{2}$. Therefore the sum of the vertex angles of the quadrilateral $(\sigma_1(-t), \sigma_1(t), \sigma_2(t), \sigma_2(-t))$ is 2π. Thus this quadrilateral is isometric to $[-t, t] \times [0, D]$ with its Euclidean metric. Letting $t \longrightarrow \infty$ yields the result. $\square$

Corollary 5.19. *Let σ be a geodesic in M and let $P(\sigma)$ be the union of all geodesics in M that are parallel to σ. Then $P(\sigma)$ is a closed convex subset of M. Moreover, $P(\sigma)$ splits isometrically as a product $Q \times \mathbb{R}$, where Q is closed and convex and $\{q\} \times \mathbb{R}$ is parallel to σ for all $q \in Q$.*

Proof. The convexity of $P(\sigma)$ is a direct consequence of the flat strip theorem. Now, let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in $P(\sigma)$ converging to some $x_\infty \in M$. For all n, there exists a unit speed geodesic σ_n parallel to σ such that $\sigma_n(0) = x_n$. Now, for all n, m, the geodesics σ_n and σ_m are parallel and hence the function $t \mapsto d(\sigma_n(t), \sigma_m(t))$ is constant equal to $d(\sigma_n(0), \sigma_m(0)) = d(x_n, x_m)$. Hence, for all t, the sequence $(\sigma_n(t))_{n \in \mathbb{N}}$ is a Cauchy sequence and therefore, by completeness of M, converges to a point, say $\sigma_\infty(t)$. It is now easily checked that $t \mapsto \sigma_\infty(t)$ is a geodesic in M parallel to σ. Thus $P(\sigma)$ is closed.

Let x and y be two points in $P(\sigma)$, and let $q = \sigma(0)$. Up to parametrization there is a unique unit speed geodesic σ_x, resp. σ_y, through x, resp. y, and parallel to σ. We can choose the parametrization of σ_x, resp. σ_y, so that $q_x := \sigma_x(0)$, resp. $q_y := \sigma_y(0)$, is the point on $\sigma_x(\mathbb{R})$, resp. $\sigma_y(\mathbb{R})$, closest to q.

The geodesics σ and σ_x bound a flat strip and therefore, for all $a \in \mathbb{R}$,

$$d(\sigma(t), \sigma_x(a)) - t = \left(d(\sigma(a), \sigma_x(a))^2 + (t-a)^2\right)^{\frac{1}{2}} - t \longrightarrow -a, \text{ as } t \longrightarrow +\infty \,.$$

Hence q_x, resp. q, is the only point on $\sigma_x(\mathbb{R})$, resp. $\sigma(\mathbb{R})$, so that $d(\sigma(t), q_x) - t \longrightarrow 0$ as $t \longrightarrow \infty$, resp. $d(\sigma_x(t), q) - t \longrightarrow 0$ as $t \longrightarrow \infty$.

Now, $d(\sigma_y(t), q_x) - t \leq d(\sigma_y(t), \sigma(t/2)) - \frac{t}{2} + d(\sigma(t/2), q_x) - \frac{t}{2}$ and since $d(\sigma_y(t), \sigma(t/2)) - \frac{t}{2} \longrightarrow 0$ as $t \longrightarrow \infty$, we get

$$\lim_{t \longrightarrow \infty} d(\sigma_y(t), q_x) - t = 0$$

and, similarly,

$$\lim_{t \longrightarrow \infty} d(\sigma_x(t), q_y) - t = 0$$

Since σ_x and σ_y are parallel, they bound a flat strip and therefore q_x, resp. q_y, is the point on $\sigma_x(\mathbb{R})$, resp. $\sigma_y(\mathbb{R})$, closest to q_y, resp. q_x. Hence,

$$d(x,y)^2 = d(q_x,q_y)^2 + \big(d(y,q_y) - d(x,q_x)\big)^2 ,$$

thus the result with $Q = \{q_x,\ x \in P(\sigma)\}$. $\square$

5.2.3. *The boundary at infinity.*

Let M be a Hadamard manifold.

Definition 5.20. *Two (unit speed) geodesics rays $\sigma, \tau : [0,+\infty) \longrightarrow M$ are called asymptotic if the function $t \mapsto d(\sigma(t),\tau(t))$ is bounded.*

Definition 5.21. *The boundary at infinity $\partial_\infty M$ of M is the set of equivalence classes of rays for the equivalence relation "being asymptotic". The equivalence class of a ray σ will be denoted $\sigma(\infty)$.*

It follows from the results in the previous section that if σ and τ are two asymptotic geodesic rays, then $\sphericalangle_{\sigma(0)}(\sigma(1),\tau(0)) + \sphericalangle_{\tau(0)}(\tau(1),\sigma(0)) \le \pi$ with equality if and only if σ and τ bound a flat half strip, namely a region isometric to $[0,D] \times [0,+\infty)$, where $D = d(\sigma(0),\tau(0))$.

The distance function $t \mapsto d(\sigma(t),\tau(t))$ between two rays σ and τ is convex and therefore two asymptotic rays cannot have a point in common unless they are equal. Hence, given a point $x \in M$, the map Φ_x from the unit sphere $U_xM \subset T_xM$ into $\partial_\infty M$ given by $\Phi_x(v) = \gamma_v(\infty)$ is injective.

If σ is geodesic a ray and x a point in M, call γ_n the geodesic ray starting at x and passing through $\sigma(n)$, $n \in \mathbb{N}$. Comparison with Euclidean triangles shows that $\sphericalangle_{\sigma(n)}(\sigma(0),x) \longrightarrow 0$ as $n \longrightarrow +\infty$ since $d(\sigma(0),\sigma(n)) \longrightarrow +\infty$. Hence $\sphericalangle_{\sigma(n)}(x,\sigma(n+k)) \longrightarrow \pi$ as $n \longrightarrow +\infty$ uniformly on k so that $\sphericalangle_x(\sigma(n),\sigma(n+k)) \longrightarrow \pi$ as $n \longrightarrow +\infty$ uniformly on k. This implies that for all $t \ge 0$, $(\gamma_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges to a point that we call $\gamma(t)$. The curve $t \mapsto \gamma(t)$ is easily seen to be a geodesic ray in M. Now, $d(\gamma(t),\sigma(t)) \le d(\gamma(t),\gamma_n(t)) + d(\gamma_n(t),\sigma(t))$. For n large enough, $d(\gamma(t),\gamma_n(t))$ is small whereas $d(\gamma_n(t),\sigma(t))$ is bounded by $d(x,\sigma(0))$. Hence $t \mapsto d(\gamma(t),\sigma(t))$ is bounded and γ is asymptotic to σ.

Thus, for all $x \in M$, $\Phi_x : U_xM \longrightarrow \partial_\infty M$ is a bijective map.

Given $x \in M$, the bijection Φ_x allows to define a distance $\sphericalangle_x$ on $\partial_\infty M$ as follows : if ξ and η are two point at infinity, then $\sphericalangle_x(\xi,\eta)$ is the distance in U_xM of the vectors u and v such that $\sigma_u(\infty) = \xi$ and $\sigma_v(\infty) = \eta$. This metric defines a topology on $\partial_\infty M$. The following lemma shows that this topology is in fact independent of the point x. It is called the *cone topology.*

Lemma 5.22. *Let x and y be two points in M. The map $\Phi_y^{-1} \circ \Phi_x : U_xM \longrightarrow U_yM$ is a homeomorphism.*

Proof. Let (u_n) be a sequence of unit tangent vectors at x, converging to some $u \in U_xM$. Let $\sigma_n : t \mapsto \exp_x(tu_n)$ and $\sigma : t \mapsto \exp_x(tu)$ be the corresponding geodesic rays. Let now v_n and v be the unit tangent vectors at y such that the geodesic rays $\gamma_n : t \mapsto \exp_y(tv_n)$ and $\gamma : t \mapsto \exp_y(tv)$ satisfy $\gamma_n(\infty) = \sigma_n(\infty)$ and $\gamma(\infty) = \sigma(\infty)$. We want to prove that the sequence (v_n) converges to v in U_yM, namely that $\sphericalangle_y(\sigma_n(\infty),\sigma(\infty)) \longrightarrow 0$ as $n \longrightarrow \infty$.

For $k \in \mathbb{N}$,

$$\sphericalangle_y(\sigma_n(\infty),\sigma(\infty)) \le \sphericalangle_y(\sigma_n(\infty),\sigma_n(k)) + \sphericalangle_y(\sigma_n(k),\sigma(k)) + \sphericalangle_y(\sigma(k),\sigma(\infty)) .$$

Moreover, $\sphericalangle_y(\sigma_n(\infty),\sigma_n(k)) \le \pi - \sphericalangle_{\sigma_n(k)}(y,\sigma_n(\infty)) = \sphericalangle_{\sigma_n(k)}(x,y)$. Clearly, if x_0 and y_0 are two points in Euclidean 2-space and if p_k is a point at distance k from x_0, then $\sphericalangle_{p_k}(x_0,y_0) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore $\sphericalangle_{\sigma_n(k)}(x,y) \longrightarrow 0$ as $k \longrightarrow \infty$, uniformly on n.

Similarly, $\measuredangle_y(\sigma(k),\sigma(\infty)) \leq \pi - \measuredangle_{\sigma(k)}(y,\sigma(\infty)) = \measuredangle_{\sigma(k)}(x,y) \longrightarrow 0$ as $k \longrightarrow \infty$.

Therefore, given $\varepsilon > 0$, we can find k so that $\measuredangle_y(\sigma_n(\infty),\sigma(\infty)) < 2\varepsilon + \measuredangle_y(\sigma_n(k),\sigma(k))$. Now, the sequence $(\sigma_n(k))_{n\in\mathbb{N}}$ converges to $\sigma(k)$, hence, for n big enough, $\measuredangle_y(\sigma_n(k),\sigma(k)) < \varepsilon$ and the result follows. $\square$

The union $\overline{M} := M \cup \partial_\infty M$ can also be given a topology extending both the topology of M and of $\partial_\infty M$: a basis of open sets is given by

- the open metric balls in M, and
- the sets $W(m,\xi,r,\varepsilon) := \{x \in \overline{M} \mid \measuredangle_m(\sigma_{mx}(\infty),\xi) < \varepsilon\} \backslash B(m,r)$, where $m \in M$, $\xi \in \partial_\infty M$, $r > 0$, $\varepsilon > 0$, and σ_{mx} denotes the geodesic ray starting from m and passing through x.

With this topology, $\overline{M}$ is homeomorphic to a closed ball.

It should be noted that the isometries of M act by homeomorphisms on $\overline{M}$ and $\partial_\infty M$.

6. Symmetric spaces of non-compact type

We now apply what we saw in the preceding sections to symmetric spaces of non-compact type. We try to give geometric proofs of some algebraic results. Our exposition follows quite closely [E, chap. 2].

6.1. **Definition and first properties.**

Definition 6.1. *A Riemannian symmetric space (M,g) is said to be of non-compact type if it is non-positively curved and if it has no Euclidean de Rham local factor (i.e. the universal cover of M does not split isometrically as $\mathbb{R}^k \times N$).*

Example. It follows from Proposition 4.12 that our favorite symmetric space $M = P(n,\mathbb{R})$ is non-positively curved. However, it is not a symmetric space of non-compact type since it does split isometrically as $\mathbb{R} \times M_1$, where $M_1 = P_1(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is the space of positive-definite symmetric matrices of determinant 1. M_1 is a symmetric space of non-compact type. The Lie algebra of its isometry group $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ admits the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \otimes \mathfrak{k}$, where $\mathfrak{p}$ is the space of trace free symmetric matrices and $\mathfrak{k}$ the space of skew-symmetric matrices.

Proposition 6.2. *A Riemannian symmetric space of non-compact type M is simply-connected (and therefore diffeomorphic to $\mathbb{R}^{\dim M}$).*

Proof. Let M be a symmetric space of non-compact type and assume that M is not simply connected. Let Γ be its fundamental group and $\pi : \widetilde{M} \longrightarrow M$ be its universal cover, so that $M = \widetilde{M}/\Gamma$. Then $\widetilde{M}$ is symmetric. Call G the identity component of its isometry group and $Z(\Gamma)$ the centralizer of Γ in G.

We claim that $Z(\Gamma)$ is transitive on $\widetilde{M}$. Indeed, let x and y be two points of $\widetilde{M}$ and choose f in the identity component of the isometry group of M such that $f(\pi(x)) = \pi(y)$. Then, $f \circ \pi : \widetilde{M} \longrightarrow M$ is a Riemannian covering. We can lift $f \circ \pi$ to a map $F : \widetilde{M} \longrightarrow \widetilde{M}$ such that $F(x) = y$ and $\pi \circ F = f \circ \pi$. F is a local isometry between complete manifolds, hence a Riemannian covering, hence an isometry since $\widetilde{M}$ is simply connected. Therefore $F \in G$ (since we can also lift homotopies).

For $\gamma \in \Gamma$, $\pi \circ F \circ \gamma = f \circ \pi \circ \gamma = f \circ \pi = \pi \circ F$. Hence there exists γ' in Γ such that $F \circ \gamma = \gamma' \circ F$, i.e. F belongs to the normalizer $N(\Gamma)$ of Γ in G and this normalizer is transitive on $\widetilde{M}$. Thus the identity component of $N(\Gamma)$, which centralizes Γ (since Γ is discrete), is still transitive on $\widetilde{M}$.

This implies that the elements of Γ are *Clifford translations*, namely, that their displacement function is constant on $\widetilde{M}$ (i.e. $\forall\gamma\in\Gamma$, $\forall x,y\in\widetilde{M}$, $d(x,\gamma x)=d(y,\gamma y)$). For if x and y are in $\widetilde{M}$ and if $z\in Z(\Gamma)$ is such that $zx=y$, then for all $\gamma\in\Gamma$, $d(y,\gamma y)=d(zx,\gamma zx)=d(zx,z\gamma x)=d(x,\gamma x)$.

Let now $\gamma\in\Gamma$ and $x\in\widetilde{M}$. Call σ the geodesic from x to γx. Then σ is γ-invariant since $\gamma x\in\sigma\cap\gamma\sigma$ and, γ being a Clifford translation, σ and $\gamma\sigma$ are parallel. γ acts on σ by translation.

Pick a point y in $\widetilde{M}$ and consider the geodesic $z\sigma$, where $z\in Z(\Gamma)$ is such that $zx=y$. Then

$$d(z\sigma(t),\sigma(t))=d(\gamma z\sigma(t),\gamma\sigma(t))=d(z\gamma\sigma(t),\gamma\sigma(t))=d(z\sigma(t+\delta),\sigma(t+\delta))$$

and therefore, if $\gamma\neq\mathrm{Id}$, the function $t\mapsto d(z\sigma(t),\sigma(t))$ is periodic and hence bounded (since continuous). Thus $z\sigma$ is parallel to σ and we have shown that every point of $\widetilde{M}$ belongs to a geodesic parallel to σ.

Corollary 5.19 then implies that $\widetilde{M}$ has a non trivial Euclidean de Rham factor. Contradiction. $\square$

Using the same kind of ideas, one proves

Theorem 6.3. [E, p. 69] *The identity component G of the isometry group of a symmetric space M of non-compact type is semi-simple and has trivial center.*

Proof. By contradiction. If G is not semi-simple then there are non-trivial connected normal Abelian Lie subgroup of G. Let A be such a subgroup. We can assume that the Lie algebra $\mathfrak{a}$ of A is maximal, i.e. not properly contained in a bigger Abelian ideal of the Lie algebra $\mathfrak{g}$ of G. Let $m\in M$, s the geodesic symmetry at m and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. We claim that $\mathfrak{a}\cap\mathrm{Ad}(s)\mathfrak{a}\neq\{0\}$. Indeed, if not, then $\mathrm{Ad}(s)\mathfrak{a}$ is also an Abelian ideal and so is $\mathfrak{a}\oplus\mathrm{Ad}(s)\mathfrak{a}$, which properly contains $\mathfrak{a}$. Therefore, $\mathfrak{b}:=\mathfrak{a}\cap\mathrm{Ad}(s)\mathfrak{a}$ is a non-trivial $\mathrm{Ad}(s)$-invariant Abelian ideal of $\mathfrak{g}$. Hence $\mathfrak{b}=(\mathfrak{b}\cap\mathfrak{k})\oplus(\mathfrak{b}\cap\mathfrak{p})$. Now, $\mathfrak{b}\cap\mathfrak{p}\neq\{0\}$. For if $\mathfrak{b}\subset\mathfrak{k}$, then on the one hand $[\mathfrak{b},\mathfrak{p}]\subset\mathfrak{b}\subset\mathfrak{k}$ because $\mathfrak{b}$ is an ideal, and on the other hand $[\mathfrak{b},\mathfrak{p}]\subset[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, hence $[\mathfrak{b},\mathfrak{p}]=0$ which implies $\mathfrak{b}=0$ since the linear isotropy representation of $\mathfrak{k}$ is faithful. We conclude that A contains a 1-parameter subgroup of transvections $t\mapsto p_t$ along some geodesic $\gamma: t\mapsto p_t(m)$.

Assume that some $\eta\in\partial_\infty M$ can be joined to $\gamma(+\infty)$ by a geodesic, say σ: $\sigma(+\infty)=\eta$ and $\sigma(-\infty)=\gamma(+\infty)$. Call $t\mapsto q_t$ the 1-parameter group of transvections along σ.

For any $x\in M$, we have

$$\sphericalangle_{\sigma(0)}(q_tx,\eta)=\sphericalangle_{q_t^{-1}\sigma(0)}(x,q_t^{-1}\eta)=\sphericalangle_{\sigma(-t)}(x,\eta)=\sphericalangle_{\sigma(-t)}(x,\sigma(0))\longrightarrow 0 \text{ as } t\longrightarrow+\infty\ ,$$

hence $q_tx\longrightarrow\eta$ as $t\longrightarrow+\infty$. Moreover,

$$\begin{aligned}\sphericalangle_m(q_t\gamma(-\infty),\eta) &\leq \sphericalangle_m(q_t\gamma(-\infty),q_tm)+\sphericalangle_m(q_tm,\eta)\\ &\leq \sphericalangle_{q_t^{-1}m}(\gamma(-\infty),m)+\sphericalangle_m(q_tm,\eta)\\ &\leq \pi-\sphericalangle_m(q_t^{-1}m,\gamma(-\infty))+\sphericalangle_m(q_tm,\eta)\\ &\leq \sphericalangle_m(q_t^{-1}m,\gamma(+\infty))+\sphericalangle_m(q_tm,\eta)\end{aligned}$$

Since $\gamma(+\infty)=\sigma(-\infty)$, we get $\sphericalangle_m(q_t\gamma(-\infty),\eta)\longrightarrow 0$ as $t\longrightarrow+\infty$.

This implies that η is in the closure of the orbit of $\gamma(-\infty)$ under the group G. Denote by $\Lambda(A)$ the set of cluster points in $\partial_\infty M$ of the orbit $A.x$ of some point $x\in M$ under A. The subset $\Lambda(A)$ is closed and independent of the choice of the point x. Since $p_t^{-1}m\longrightarrow\gamma(-\infty)$,

$\gamma(-\infty) \in \Lambda(A)$. The subgroup A being normal in G, $\Lambda(A)$ is stable by G and therefore $\eta \in \overline{\Lambda(A)} = \Lambda(A)$. Now, the fact that A is Abelian implies that A fixes $\Lambda(A)$ pointwise. Hence for all t, $p_t\eta = \eta$. But the proof above shows that $p_{-t}\eta \longrightarrow \gamma(-\infty)$ as $t \longrightarrow +\infty$. Thus $\eta = \gamma(-\infty)$.

We have proved that every point in M belongs to a geodesic joining $\gamma(-\infty)$ to $\gamma(+\infty)$, hence parallel to γ. Corollary 5.19 then implies that M has a non trivial Euclidean de Rham factor. Contradiction.

Assume now that A is a discrete Abelian normal subgroup in G. Take $a \in A$ and $x \in M$. For each $y \in M$ there exists $g \in G$ such that $y = gx$. Therefore $d(y, ay) = d(gx, agx) = d(x, g^{-1}agx) = d(x, ax)$ since A being discrete and G connected, G actually centralizes A. The contradiction follows as in the proof of the previous proposition, hence G has trivial center. $\square$

Concerning the action on $\partial_\infty M$ of the identity component G of the isometry group of M, we have:

Proposition 6.4. [E, pp. 59 & 101] *Let $\xi \in \partial_\infty M$, $m \in M$ and let K be the isotropy subgroup of G at m. Then $G.\xi = K.\xi$. Moreover, the stabilizer G_ξ of ξ in G acts transitively on M.*

Remark 6.5. This property is a weak geometric version of the Iwasawa decomposition of non-compact semisimple Lie groups. The full geometric version of the latter decomposition requires to introduce horospheric coordinates.

Proof. Call γ the geodesic ray emanating from m and belonging to ξ and $t \mapsto p_t$ the 1-parameter group of transvections along this ray.

Let $g \in G$. We want to prove that there exists $k \in K$ so that $k\xi = g\xi$. Call σ_t the geodesic ray starting from m and passing through the point gp_tm and set $\xi_t = \sigma_t(\infty)$. Let q_t be the transvection along σ_t such that $q_tm = gp_tm$. Note that $q_t\xi_t = \xi_t$.

The isometry $k_t := q_t^{-1}gp_t$ belongs to K. Moreover,

$$\sphericalangle_m(k_t\xi, \xi_t) = \sphericalangle_{q_tm}(gp_t\xi, q_t\xi_t) = \sphericalangle_{gp_tm}(g\xi, \xi_t) = \sphericalangle_{gp_tm}(m, gm)$$

and this last quantity goes to 0 as t goes to ∞. Similarly,

$$\sphericalangle_m(\xi_t, g\xi) = \sphericalangle_m(gp_tm, g\xi) \leq \pi - \sphericalangle_{gp_tm}(m, g\xi) = \sphericalangle_{gp_tm}(m, gm) \longrightarrow 0\ .$$

Hence $k_t\xi \longrightarrow g\xi$ as $t \longrightarrow \infty$. Since K is compact, there exists $k \in K$ so that $k\xi = g\xi$ as wanted.

Now let m' be another point of M and let $g \in G$ be such that $gm = m'$. It follows from what we just proved that there exists $k \in K$ so that $k\xi = g^{-1}\xi$. Now $gk\xi = \xi$ and $gkm = m'$. Therefore G_ξ is transitive on M. $\square$

Example. For $M_1 = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$, the points at infinity can be identified with eigenvalues-flag pairs, as follows: For $\xi \in \partial_\infty M_1$, there is a unique $X \in \mathfrak{p}$ (namely, a trace free symmetric matrix) of norm one such that $\xi = \gamma_X(+\infty)$, where $\gamma_X(t) = e^{tX}$. Call $\lambda_i(\xi)$ the distinct eigenvalues of X arranged so that $\lambda_1(\xi) > \ldots > \lambda_k(\xi)$, and let $E_i(\xi)$ be the corresponding eigenspaces. Put $V_i(\xi) = \bigoplus_{j \leq i} E_j(\xi)$. To the point ξ, we have therefore associated a vector $\lambda(\xi) = (\lambda_1(\xi), \ldots, \lambda_k(\xi))$ and a flag $V(\xi) = (V_1(\xi) \subset \ldots \subset V_k(\xi))$ of $\mathbb{R}^n$ such that

- $\lambda_1(\xi) > \ldots > \lambda_k(\xi)$,
- $\sum_i (\dim V_i(\xi) - \dim V_{i-1}(\xi))\lambda_i(\xi) = 0$ (since X is trace free),
- $\sum_i (\dim V_i(\xi) - \dim V_{i-1}(\xi))\lambda_i(\xi)^2 = 1$ (since X has norm 1).

Conversely, it is easily seen that given a vector $\lambda = (\lambda_1, \dots, \lambda_k)$ and a flag $V = (V_1 \subset \dots \subset V_k)$ satisfying those conditions, there is a unique point $\xi \in \partial_\infty M_1$ such that $\lambda(\xi) = \lambda$ and $V(\xi) = V$.

One can also check that the action of $g \in \mathrm{SL}(n, \mathbb{R})$ on the eigenvalues-flag pairs corresponding to its action on $\partial_\infty M_1$ is given by $g.(\lambda, V) = (\lambda, gV)$ where gV is the flag $gV_1 \subset \dots \subset gV_k$.

6.2. **Totally geodesic subspaces.**

A submanifold N of (M, g) is said to be totally geodesic if the Levi-Civitá connection of the metric on N induced by g is simply the restriction of the Levi-Civitá connection of g. This means that any geodesic γ of M such that $\gamma(0) \in N$ and $\dot\gamma(0) \in T_{\gamma(0)}N$ stays in N.

Let N be a totally geodesic submanifold of M and let $m \in N$. Then necessarily, for any tangent vectors u, v, w to N at m, $R(u, v)w$ is also tangent to N at m (since R is also the curvature tensor of the induced metric on N). If we consider the Cartan decomposition of $\mathfrak{g}$ associated to m, this means that, if we see T_mN as a subspace $\mathfrak{q}$ of $\mathfrak{p}$, $[[\mathfrak{q}, \mathfrak{q}], \mathfrak{q}] \subset \mathfrak{q}$. Such a $\mathfrak{q}$ is called a *Lie triple system*.

Conversely, if $\mathfrak{q} \subset \mathfrak{p}$ is a Lie triple system, then the (complete) manifold $e^{\mathfrak{q}}m$ is totally geodesic. Indeed, one checks that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] + \mathfrak{q}$ is a subalgebra of $\mathfrak{g}$. If H is the analytic subgroup of G whose Lie algebra is $\mathfrak{h}$ then let N be the orbit $H.m$. Clearly, a geodesic tangent to N at m is of the form $t \mapsto e^{tX}m$ with $X \in \mathfrak{q}$. Hence a geodesic through $x \in N$ is of the form $t \mapsto he^{tX}m$ with $X \in \mathfrak{q}$ and $h \in H$ such that $hm = x$, thus is contained in N. Hence N is totally geodesic. Now any point x of N can be joined to m by a geodesic inside N, hence $N = e^{\mathfrak{q}}m = \exp_m(T_mN)$.

6.3. **Flats.**

Definition 6.6. *A k-flat F in M is a complete totally geodesic submanifold of M isometric to a Euclidean space $\mathbb{R}^k$.*

Obviously, if F is a k-flat of M, then $1 \le k \le \mathrm{dim} M$.

Definition 6.7. *The rank $r = \mathrm{rk}(M)$ of the symmetric space M is defined to be the maximal dimension of a flat in M. A r-flat is therefore a flat of maximal dimension.*

Proposition 6.8. *The flats through $m \in M$ are in one-to-one correspondence with Abelian subspaces of $\mathfrak{p} = T_mM$. Moreover, if $\mathfrak{a}$ is such an Abelian subspace (seen as a subspace of T_mM), then $\exp_m : \mathfrak{a} \longrightarrow F := \exp_m(\mathfrak{a})$ is an isometry.*

Proof. The first assertion is a direct consequence of the curvature formula and the discussion about totally geodesic submanifolds of M. Now let $A \in \mathfrak{a}$ seen as a subspace of T_mM and $\xi \in T_A\mathfrak{a} = \mathfrak{a}$. Then

$$\mathrm{d}_A \exp_m(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} \exp_m(A + t\xi)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{A+t\xi}m|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{A}e^{t\xi}m|_{t=0} = \mathrm{d}_m e^A(\xi)\ .$$

Since e^A is an isometry, $\|\mathrm{d}_A \exp_m(\xi)\|_{\exp_m(A)} = \|\xi\|_m$. □

Example. For $M_1 = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$, a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = T_{\mathrm{id}}M$ is the space of trace free diagonal matrices. Therefore, the rank of M_1 is $n - 1$.

The identity component G of the isometry group of the symmetric space M in general does not act transitively on the tangent bundle TM of M (nor on geodesics in M), but it acts transitively on the pairs (x, F), where x is a point in M and F a r-flat through x. Indeed:

Theorem 6.9. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G, and let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal Abelian subspaces of $\mathfrak{p}$. Then there exists $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}'$.*

Proof. See P.-E. Paradan's lecture [P]. $\square$

In particular, any geodesic of M is contained in a maximal flat.

6.4. **Regular geodesics. Weyl chambers.**
We refer to [E, p. 85-94] for details.

A geodesic of M is called *regular* if it is contained in a unique maximal flat. Otherwise, it is called *singular*. In the same way, a tangent vector $v \in T_mM$ (or the corresponding element of $\mathfrak{p}$) is defined to be regular, resp. singular, if the geodesic $\gamma_v : t \mapsto \exp_m(tv)$ is regular, resp. singular.

Example. For our symmetric space $M_1 = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, an element of $\mathfrak{a}$ (that is, a diagonal matrix of trace zero) is regular if and only if its coefficients are all distinct.

If a geodesic γ, resp. a tangent vector v, is regular, we denote by $F(\gamma)$, resp. $F(v)$, the unique maximal flat containing γ, resp. γ_v.

If σ and τ are two asymptotic rays, it follows from Proposition 6.4 that σ is regular if and only if τ is. Therefore we may define a point $\xi \in \partial_\infty M$ to be regular if some (hence any) ray belonging to ξ is regular.

If v is a unit tangent vector at some point $m \in M$ and if x is a point in M, we call $v(x)$ the unit tangent vector at x asymptotic to v, namely, such that $\gamma_{v(x)}(+\infty) = \gamma_v(+\infty)$. Note that if v is regular, then $v(x)$ is regular for all $x \in M$.

We will define three kinds of Weyl chambers: in the tangent bundle TM (or the unit tangent bundle UM) of M, in M itself, and on the boundary at infinity of M.

Let v_0 and v_1 be two regular (unit) tangent vectors at a point $m \in M$. Call v_0 and v_1 equivalent if there is a flat F through m and a curve $t \mapsto v(t)$ of regular (unit) tangent vectors at m, joining v_0 to v_1, and tangent to F for all t. The equivalence classes for this equivalence relation on the regular vectors in T_mM (in U_mM) are called *Weyl chambers at m*. Given a regular vector $v \in T_mM$ (or U_mM), we call $\mathcal{C}(v)$ the Weyl chamber of v.

Example. There are therefore $n!$ Weyl chambers in the maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ for $M_1 = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$: if A is a diagonal matrix with distinct coefficients $a_1, \dots, a_n$, there exists a permutation τ such that $a_{\tau(1)} > \dots > a_{\tau(n)}$ and the Weyl Chamber of A is the set of diagonal matrices $A' = \mathrm{diag}(a'_1, \dots, a'_n)$ such that $a'_{\tau(1)} > \dots > a'_{\tau(n)}$.

If $\mathcal{C} \subset U_mM$ is a Weyl chamber, we define its *center* to be the unit vector at m pointing in the same direction as $\int_{\mathcal{C}} \iota(u) d\mu_S(u)$, where $S \subset U_mM$ is the great subsphere of smallest dimension containing $\mathcal{C}$, μ_S is Lebesgue measure on S, and $\iota : U_mM \longrightarrow T_mM$ is the inclusion.

Let $v \in U_mM$ be a regular vector and $F(v)$ the corresponding maximal flat. We define the *Weyl chamber of v in $F(v)$* as follows:

$$W(v) = \{\exp_m(tu) |\ u \in \mathcal{C}(v),\ t > 0\}\ .$$

One proves that the singular geodesics through a point m in a maximal flat F form the union of a finite number of hyperplanes in F, called *walls*, and the connected components of $F \backslash \{\text{walls}\}$ are precisely the Weyl chambers $W(v)$ for v regular unit tangent vectors to F at m.

We now give, without proof, some of the most important properties of Weyl chambers (see [E]).

(1) If $v \in U_m M$ is a regular vector and if $F(v)$ is the maximal flat through m tangent to v, then $W(v)$ is an open unbounded convex subset of $F(v)$.
(2) If $v \in U_m M$ is a regular vector and if $x \in M$, then the Weyl chambers $W(v)$ and $W(v(x))$ are asymptotic, more precisely, the Hausdorff distance between them is bounded by the distance between m and x.
(3) For $v \in U_m M$ and $v' \in U_{m'} M$ two regular vectors, there exists an element $g \in G$ such that $gm = m'$ and $\mathrm{d}_m g(v) \in \mathcal{C}(v')$, hence $g\,\mathcal{C}(v) = \mathcal{C}(v')$ and $g\,W(v) = W(v')$, thus implying that any two Weyl chambers are isometric.

The third kind of Weyl chambers is simply the asymptotic version of the previous ones. Let ξ be a regular point on $\partial_\infty M$ and let m and $v \in U_m M$ be such that $\gamma_v(\infty) = \xi$. Then set

$$\mathcal{C}(\xi) = \{\gamma_u(\infty) |\, u \in \mathcal{C}(v)\} \ .$$

This is well-defined by Property (2) above. Note that $\mathcal{C}(\xi)$ and $\overline{\mathcal{C}(\xi)}$ are subsets of the boundary at infinity of $F(v)$.

We say that a regular point $\xi \in \partial_\infty M$ is the center of its Weyl chamber $\mathcal{C}(\xi)$ if $\xi = \gamma_v(\infty)$ for some $v \in UM$ center of its Weyl chamber $\mathcal{C}(v)$

6.5. **Dichotomy between rank 1 and higher rank symmetric spaces.**

There are many differences, which have very important implications (for example for lattices), between symmetric spaces of non-compact type of rank 1 and of rank at least 2. Here we list only straightforward consequences of what we have seen.

Proposition 6.10. *Let M be a symmetric space of non-compact type. The following assertions are equivalent:*

(1) M has rank 1;

(2) M has strictly negative sectional curvatures (hence there exist $b > a > 0$ such that the sectional curvatures of M are pinched between $-b^2$ and $-a^2$);

(3) The isotropy group of G at some point $m \in M$ is transitive on the unit tangent vectors at m;

(4) any two points on the boundary at infinity of M can be joined by a geodesic.

Proof. (2) obviously implies (1). Conversely, assume that $u, v \in T_m M$ are such that $R_m(u, v, u, v) = 0$. Then $R_m(u, v)u = 0$ because $v \mapsto R_m(u, v)u$ is negative semi-definite. Hence $[[u, v], u] = 0$, i.e. $(\mathrm{ad}u)^2 v = 0$. Now, $\mathrm{ad}u$ is symmetric w.r.t. the bilinear form B^θ (see [P]). Thus $\mathrm{Ker}(\mathrm{ad}u)^2 \subset \mathrm{Ker}(\mathrm{ad}u)$ and $[u, v] = 0$, namely u and v are tangent to a maximal flat through m.

(1) implies (3) by Theorem 6.9. Conversely, if the rank of M is greater than 1, then a singular geodesic can not be sent to a regular one.

Assume (2). The fact that the sectional curvatures of M are bounded from above by a strictly negative constant $-a^2$ implies that geodesic triangles in M are thinner than their comparison triangles in M_{-a^2}, the 2-dimensional model space of constant curvature $-a^2$ (in other words, M is $\mathrm{CAT}(-a^2)$). Let ξ and η be two points on $\partial_\infty M$ and let σ and τ be two geodesic rays starting from some point $x \in M$ such that $\sigma(\infty) = \xi$ and $\tau(\infty) = \eta$. The distance between x and the geodesic segment $[\sigma(n), \tau(n)]$ is bounded independently of $n \in \mathbb{N}$ (because this is true in M_{-a^2}). Hence it will be possible to find a convergent subsequence and this will be the geodesic joining ξ to η: (2) implies (4).

We prove that (4) implies (1) by contradiction: assume there exists a 2-dimensional flat F in M, and choose points x and y on the boundary at infinity of F that cannot be joined by a geodesic in F. Then x and y can not be joined by a geodesic in M. Indeed, if γ is such a geodesic, m a point of F, and σ, τ the geodesic rays emanating from m such that $\sigma(\infty) = x$ and $\tau(\infty) = y$, then the Hausdorff distance between $\gamma(R)$ and $\sigma(\mathbb{R}_+) \cup \tau(\mathbb{R}_+)$ is bounded by some $k > 0$. Now, the intersection of F with the k-neighborhood of $\gamma(\mathbb{R})$ is convex and contained in the $2k$-neighborhood of $\sigma(\mathbb{R}_+) \cup \tau(\mathbb{R}_+)$. Hence, for all $n \in \mathbb{N}$, the geodesic segment $[\sigma(n), \tau(n)]$ is contained in the $2k$-neighborhood of $\sigma(\mathbb{R}_+) \cup \tau(\mathbb{R}_+)$. This is possible only if the angle between σ and τ at m is π, i.e. if x and y are joined by a geodesic inside F. □

Example. Using the eigenvalues-flag pair description of the boundary at infinity of $M_1 = P_1(n, \mathbb{R})$, one can prove (see [E, p. 93]) that two points ξ and η on $\partial_\infty M_1$ corresponding to the eigenvalues-flag pairs $((\lambda_i(\xi))_{1 \le i \le k}, (V_i(\xi))_{1 \le i \le k})$ and $((\lambda_i(\eta))_{1 \le i \le l}, (V_i(\eta))_{1 \le i \le l})$ can be joined by a geodesic if and only if

- $k = l$,
- $\forall i$, $\lambda_i(\eta) = -\lambda_{k-i+1}(\xi)$,
- $\forall i$, $\mathbb{R}^n$ is the direct sum of $V_i(\xi)$ and $V_{k-i}(\eta)$.

6.6. **Towards the building structure of the boundary at infinity.**

We just saw that in rank one symmetric spaces, two points at infinity can always be joined by a geodesic. In higher rank symmetric spaces, they can be joined by flats. A much stronger result is true: the boundary at infinity of a symmetric space of non-compact type admits the structure of a building whose apartments are the boundaries at infinity of the maximal flats (see the lecture of G. Rousseau [R] for the definition of a building).

Here we will only prove the following

Theorem 6.11. *Let M be a symmetric space of non-compact type. Any two points on the boundary at infinity $\partial_\infty M$ of M lie on the boundary at infinity $\partial_\infty F$ of a maximal flat F of M.*

Proof (adapted from [BS]).

Let n be the dimension of M and r its rank. We may assume that $r \ge 2$.

Let ξ_0 and η_0 be two points of $\partial_\infty M$. Let ξ and η be regular points of $\partial_\infty M$ so that $\xi_0 \in \overline{\mathcal{C}(\xi)}$ and $\eta_0 \in \overline{\mathcal{C}(\eta)}$. We can assume that η is the center of its Weyl chamber. It is enough to prove that there exists a flat F such that ξ and η belong to $\partial_\infty F$.

Let m be a point of M and $v \in U_m M$ so that $\gamma_v(-\infty) = \xi$. Note that γ_v is a regular geodesic. Let ϕ be a transvection along γ_v and $F(v)$ be the unique maximal flat containing γ_v. The boundary at infinity $\partial_\infty F(v)$ of $F(v)$ is the union of a finite number of Weyl chambers which are permuted by ϕ. Up to taking a power of ϕ, we can assume that ϕ fixes the centers of the Weyl chambers in $\partial_\infty F(v)$.

We claim that, up to extraction of a subsequence, the sequence $(\phi^j \eta)_{j \in N}$ converges to some point $\eta' \in \partial_\infty F(v)$. Indeed, for all $x \in \partial_\infty M$,

$$\sphericalangle_m(\phi x, \gamma_v(+\infty)) = \sphericalangle_m(\phi x, \phi m) \le \pi - \sphericalangle_{\phi m}(\phi x, m) = \pi - \sphericalangle_m(x, \phi^{-1} m) = \sphericalangle_m(x, \gamma_v(+\infty))$$

with equality if and only if the triangle $T(m, \phi m, \phi x)$ is flat, i.e. if and only if $x \in \partial_\infty F(v)$, since v is regular and $\partial_\infty F(v)$ is invariant by ϕ. Now, if y is any limit point of $\{\phi^j x,\ j \in \mathbb{N}\}$, we have $\sphericalangle_m(\phi y, \gamma_v(+\infty)) = \sphericalangle_m(y, \gamma_v(+\infty))$, hence $y \in \partial_\infty F(v)$.

Let $v_j \in U_m M$ be such that $\gamma_{v_j}(\infty) = \phi^j \eta$. Since $\phi^j \eta$ is the center of its Weyl chamber, so is v_j. Now, all the Weyl chambers are isometric. Therefore, the angle $\sphericalangle_m(v_j, \text{walls of } \mathcal{C}(v_j))$ is constant and this implies that η' is regular and is the center of its Weyl chamber.

Call γ the (regular) geodesic of $F(v)$ such that $\gamma(0) = m$ and $\gamma(+\infty) = \eta'$ and let $\zeta = \gamma(-\infty)$. Again, ζ is regular and is the center of its Weyl chamber.

Let H^u be the strong unstable horosphere of $\dot\gamma(0)$. H^u is a submanifold of the unstable horosphere H of $\dot\gamma(0)$, that is, of the horosphere centered at $\zeta = \gamma(-\infty)$ and passing through $m = \gamma(0)$. H^u is (roughly) defined as follows. Through each point x of the horosphere H there is a (unique) maximal flat F_x containing the ray joining x to ζ. Consider the distribution Q of $(n-r)$-planes in TH given by $Q_x = T_xF_x{}^\perp \subset T_xH$. One proves that this distribution is integrable and H^u is defined to be the maximal integral submanifold through m. For all $x \in H^u$, $H^u \cap F_x = \{x\}$.

Consider the map $f : H^u \times \mathcal{C}(\zeta) \longrightarrow \partial_\infty M$ given by $f(m', \zeta') = \gamma_{m'\zeta'}(-\infty)$, where $\gamma_{m'\zeta'}$ is the geodesic joining m' to ζ'. This map is continuous. Moreover, it is injective. Indeed, assume that $\gamma_1 := \gamma_{m_1\zeta_1}$ and $\gamma_2 := \gamma_{m_2\zeta_2}$ satisfy $\gamma_1(-\infty) = \gamma_2(-\infty)$. Let P be the maximal flat containing γ_1. Then, since ζ_1 and ζ_2 belongs to the same Weyl chamber, $\partial_\infty P$ contains $\gamma_2(+\infty) = \zeta_2$ and $\gamma_2(-\infty) = \gamma_1(-\infty)$. Therefore (see the proof of Proposition 6.10), there is a geodesic σ in P such that $\sigma(+\infty) = \gamma_2(+\infty)$ and $\sigma(-\infty) = \gamma_2(-\infty)$. The geodesics σ and γ_1 are both contained in P and satisfy $\sigma(-\infty) = \gamma_1(-\infty)$: they must be parallel and hence $\zeta_1 = \gamma_1(+\infty) = \sigma(+\infty) = \zeta_2$. The geodesics γ_1 and γ_2 are therefore parallel, hence they bound a flat strip, and since they are regular, they both must be contained in the maximal flat P. Now P also contains the geodesic joining m_1 to ζ, and by the definition of the strong unstable horosphere H^u, the intersection of P and H^u is reduced to m_1. Hence $m_1 = m_2$ and f is injective as claimed. Since the domain and the target of f have the same dimension, f is in fact a homeomorphism from a neighborhood $U \times V$ of (m, ζ) to a neighborhood W of η'.

Since $(\phi^j\eta)_{j\in\mathbb{N}}$ converges to η', we may assume that for all j, there exists $(m_j, \zeta_j) \in U \times V$ such that $f(m_j, \zeta_j) = \phi^j\eta$. But $\phi^j\eta$ is the center of its Weyl chamber thus so is ζ_j, i.e. $\zeta_j = \zeta$ for all j. Hence for all j there exists $\gamma_j = \gamma_{m_j\zeta}$ joining ζ to $\phi^j\eta$. Since $\gamma_j \longrightarrow \gamma$ we may assume that the geodesics γ_j are regular.

Therefore, for all j, $\phi^{-j}\gamma_j$ is a regular geodesic joining ζ to η. By the flat strip theorem, these geodesics, being regular and parallel, must all lie in the same maximal flat F. Thus $\eta \in \partial_\infty F$. Now, $\phi^{-j}m_j \in F$ for all j. Since the sequence $(m_j)_{j\in\mathbb{N}}$ is bounded and $\phi^{-j}x \longrightarrow \gamma_v(-\infty)$ as $j \longrightarrow \infty$ for all $x \in M$, we have $\phi^{-j}m_j \longrightarrow \gamma_v(-\infty)$ as $j \longrightarrow \infty$. Hence $\xi = \gamma_v(-\infty)$ belongs to $\partial_\infty F$ and we are done. $\square$

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